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Minimal sets in monotone and concave skew-product semiflows II: Two-dimensional systems of differential equations ☆

Carmen Núñez [a], Rafael Obaya [a,*], Ana M. Sanz [b]

[a] Departamento de Matemática Aplicada, E. Ingenierías Industriales, Universidad de Valladolid, 47011 Valladolid, Spain

[b] Departamento de Matemática Aplicada, E.U. Ingenierías Agrarias, Universidad de Valladolid, 42004 Soria, Spain

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ABSTRACT

Recurrent nonautonomous two-dimensional systems of differential equations of ordinary, finite delay and reaction–diffusion types given by cooperative and concave vector fields define monotone and concave skew-product semiflows, whose dynamics is analyzed in this paper. A complete description of all the minimal sets in a very interesting dynamical situation is provided, and some criteria to determine its possible existence and the area on which they lie are given. Suitable examples prove the optimality of the results.

1. Introduction

Among the large number of papers devoted to the study of monotone and sublinear, concave or convex semiflows generated by families of differential equations, the non-exhaustive list of references of Krasnoselskii [16,17], Selgrade [29], Johnson [12], Hirsch [8,9], Smith [30,31], Takáç [32], Aiello et al. [1], Johnson et al. [13,14], Wu [35], Chueshov [3], Zhao [37,38], Novo et al. [21], Novo et al. [22], and Núñez et al. [25–27], contain significative theoretical results, many of them essential in the description of mathematical models of engineering, biology, economics and other branches of applied sciences.

We present in this paper the analysis of the dynamics generated by cooperative and concave two-dimensional systems of nonautonomous equations, of ordinary, finite delay and reaction–diffusion

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* Corresponding author.

E-mail addresses: carnun@wmatem.eis.uva.es (C. Núñez), rafoba@wmatem.eis.uva.es (R. Obaya), anasan@wmatem.eis.uva.es (A.M. Sanz).

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types. Under some recurrence and differentiability properties on the temporal variation of the law, the solutions generate a regular skew-product semiflow over a minimal base. Such a formulation, which includes the uniform almost periodic and uniform almost automorphic cases, allows us to apply techniques of topological and differentiable dynamics to describe the long-term behavior of the semiorbits.

The paper is the natural continuation of Núñez et al. [28], where abstract monotone, concave and C^1 semiflows over a minimal base are analyzed under the assumption of the existence of a semicontinuous subequilibrium or of a relatively compact semiorbit. A topological classification of the possible dynamical situations is there provided, showing in particular important differences between the autonomous and nonautonomous dynamics, as well as between the concave and sublinear settings. Among those possibilities, we now focus our attention on the most significative one from the point of view of the applications, and refine the results to the particular setting of two-dimensional systems: under the assumption of existence of a semicontinuous subequilibrium (or, roughly speaking, of a semicontinuous lower-solution) and of a minimal set strongly above it, we describe the behavior of the bounded semiorbits, as well as the shapes of the set of all the minimal sets and of these sets themselves.

We briefly describe the structure and main results of the paper. Section 2 starts with a precise description of the three types of nonautonomous differential equations that we consider, whose assumed properties give rise to a monotone, concave and C^1 semiflow defined by their solutions on a product space $\Omega \times X$ with a subequilibrium $a = (a_1, a_2)$ strongly below a minimal set K^1. The base Ω is often the compact metric space provided by the hull of the vector field, with the minimal flow of translation in time, and the fiber X is the strongly ordered Banach space respectively given by $\mathbb{R}^2$, $C([-1,0],\mathbb{R}^2)$ and $C(\bar{U},\mathbb{R}^2)$ in the ordinary, finite delay and parabolic equations, these last ones with bounded spatial domain $U \subset \mathbb{R}^n$ and Neumann boundary conditions. A componentwise separation property, the relation between subequilibria and differential inequalities, and the definition and main properties of the so-called marginal semiflows, which play a fundamental role throughout the paper, are also described and proved in this section. It is completed with a brief summary of two of the results of [28]: the most fundamental ones in the present analysis.

Section 3 is the longest and contains the main results of the paper. Its body is the description of the set of minimal sets $\mathcal{M}$ in $\Omega \times X$. The dynamics above the graph of a or below the set K^1 is described in detail in [28] in the simplest dynamical situation, in which K^1 is the unique minimal set strongly above a. So that in the rest of the section we assume the existence of more than one element of $\mathcal{M}$ strongly above a, which according again to [28] means the existence of infinitely many elements of the set $\mathcal{M}_{a,K^1}$ of minimal sets below K^1 and strongly above a. By combining the componentwise separation property with previous results of Jiang and Zhao [10] and Novo et al. [23], we show at the beginning of the section that the omega-limit set of any semiorbit starting strongly above the semiequilibrium is a minimal set given by a copy of the base; i.e., by the invariant graph of a continuous map from Ω to X. So that these omega-limit sets inherit the topological and dynamical structures of Ω, and hence reproduce the temporal variation of the vector field. Also a minimal set strongly above another one is a copy of the base. In particular, the elements of $\mathcal{M}_{a,K^1}$ are copies of the base strongly above a.

If K is given by the graph of $c = (c_1, c_2) : \Omega \to X$, we write $K = \{c\} = \{c_1, c_2\}$. We also write $K^1 = \{c_1^1, c_2^1\}$ and $a = (a_1, a_2)$, and define $c^\lambda = (c_1^\lambda, c_2^\lambda) = \lambda c^1 + (1-\lambda)a$ for $\lambda \in \mathbb{R}$. A labeling procedure allows as to prove that for any $K \in \mathcal{M}_{a,K^1}$ there exists $\lambda \in (0,1]$ such that one of the three following situations holds: either $K = \{c_1^\lambda, c_2\}$ with $c_2 \gg c_2^\lambda$; or $K = \{c_1, c_2^\lambda\}$ with $c_1 \gg c_1^\lambda$; or $K = \{\tilde{c}^\lambda\}$, where $\tilde{c}^\lambda = (\tilde{c}_1^\lambda, \tilde{c}_2^\lambda) : \Omega \to X$ represents a continuous function agreeing with c^λ in a residual subset of Ω. We respectively say that K is labeled by λ only on its first component, only on its second component, or on both components. As next step, we prove that one of the two following cases holds for $\mathcal{M}_{a,K^1}$: either each $\lambda \in (0,1]$ labels infinitely many of its elements, which we call the multiple-labeling case; or each $\lambda \in (0,1]$ labels exactly one, and we are in the single-labeling case. (This is an important difference with the sublinear case, for which multiple and single labels may coexist, as seen in [28].)

The section is completed with an exhaustive analysis of the whole $\mathcal{M}$ in the multiple and single-labeling cases for $\mathcal{M}_{a,K^1}$, part of which we now summarize. In the multiple-labeling case, $\mathcal{M} = \{\{c_1^{\alpha_1}, c_2^{\alpha_2}\} \mid (\alpha_1, \alpha_2) \in \mathcal{F}\}$ for a closed order-convex set $\mathcal{F} \subset \mathbb{R}^2$ containing $\{(\lambda,\lambda) \mid \lambda \in [0,1]\}$ and with nonempty interior. In addition, the subsets of its points above or below a fixed one in its interior agree with the closure of its interior. The set $\mathcal{F}$ is bounded from above if and only if there exists a top minimal set, and it is not necessarily bounded from below. The single-labeling case is completely different and more complex. As a first step, we prove that the set $\mathcal{M}_a$ of minimal sets strongly above a is given by $\{K^\lambda \mid \lambda \in J\}$ for a right-closed interval $J \subseteq (0,\infty)$ with left edge 0, and again there are three possibilities: $K^\lambda = \{\tilde{c}^\lambda\}$ for any $\lambda \in J$; $K^\lambda = \{c_1^\lambda, d_2^\lambda\}$ for any $\lambda \in J$ with $d_2^\lambda = \tilde{c}_2^\lambda$ either just for $\lambda \in J_1 = \{1\}$ or for a proper subinterval $J_1 \subset J$ containing $(0,1]$, and with $d_2^\lambda \gg c_2^\lambda$ and $d_2^\lambda \ll c_2^\lambda$ to the left and right of J_1, respectively; or $K^\lambda = \{d_1^\lambda, c_2^\lambda\}$ for any $\lambda \in J$ and properties symmetric to the previous case. After this, we extend the description to the whole $\mathcal{M}$. Here, the presence of minimal sets which are not copies of the base is possible, as we show with suitable examples. However, either $\tilde{c}_1^\lambda$ is the first component of every minimal set, where λ varies in a closed interval $J_* \subseteq \mathbb{R}$ with $J_* \cap (0,\infty) = J$, or this happens with the second component. Assume that the first case holds. Then, for each $\lambda \in J_*$ there exists $M^\lambda \in \mathcal{M}$ with $M^\lambda = K^\lambda$ for $\lambda \in J$ such that: the family $(M^\lambda)_{\lambda \in J_*}$ increases with λ; for $\lambda > \inf J_*$, $M^\lambda = \{\tilde{c}_1^\lambda, d_2^\lambda\}$, and the map $\lambda \mapsto d_2^\lambda(\omega)$ is concave for every $\omega \in \Omega$; if $\lambda_1 = \inf J_* \in \mathbb{R}$, then M^{λ_1} is an almost automorphic extension of the base (perhaps a new copy); if a set $N \in \mathcal{M}$ different from any M^λ exists, and its first component is $\tilde{c}_1^{\lambda_1}$, then its second state component is strongly below the second one of M^{λ_1}; and, in most of the cases, if two such sets N_1 and N_2 exist, they cannot be ordered. The possibility $\mathcal{M} = \{\{\tilde{c}^\lambda\} \mid \lambda \in J_*\}$ may of course hold.

In Section 4 we show that, in the above conditions and in the multiple-labeling case, there exists a region in the phase space in which the two-dimensional system uncouples in two independent scalar equations given by affine functions, with nonempty interior always in the ordinary and parabolic cases and in some interesting situations in the delay case. This assertion is also true for the first one of the equations in two situations inside the single-labeling case: if $\mathcal{M} = \{\{\tilde{c}^\lambda\} \mid \lambda \in J_*\}$ and a_2 is not continuous, or if this is not the case but any minimal set has the map $\tilde{c}_1^\lambda$ for a $\lambda \in J_*$ as its first component. And the analogous conclusion for the second equation of the system holds in the two symmetric situations. All these results can also be understood as negative criteria precluding the existence of more than one minimal set strongly above a subequilibrium. We complete this section by pointing out that some of the arguments used through these proofs can be adapted to the sublinear case analyzed in the paper [27] in order to extend the scope of the results of its Section 4.1 to the single-labeling case.

The paper is concluded by applying the previous results to the long-term analysis of a delayed Hopfield-type neural network with nonautonomous law, with and without self-connection between cells. This is the content of Section 5.

2. Two-dimensional systems of differential equations and semiflows

Let us begin this section by describing the two-dimensional systems of nonautonomous ordinary, delay and parabolic differential equations we consider, and the monotone and concave skew-product semiflows they induce. The reader is referred to [28] for the basic definitions of real continuous flows and semiflows on complete Banach spaces, orbits, semiorbits and backward orbits, omega-limit and minimal sets, almost automorphic extensions of a base, strong partial order, ordering of maps and sets, distallity, uniform stability of positively invariant sets, etc.

2.1. Two-dimensional systems and skew-product semiflows

Let $(\Omega, \sigma, \mathbb{R})$ be a real continuous flow in a compact metric space, and write $\sigma(t,\omega) = \sigma_t(\omega) = \omega \cdot t$. In the rest of the paper, we assume that Ω is minimal and consider skew-product semiflows written as

$$\tau : \mathbb{R}_+ \times \Omega \times X \longrightarrow \Omega \times X, \qquad (t,\omega,x) \mapsto \big(\omega \cdot t, u(t,\omega,x)\big), \tag{2.1}$$

for which both the fiber space X (a strongly ordered Banach space X with a *monotone* norm $\|\cdot\|$: $\|x\| \leqslant \|y\|$ whenever $0 \leqslant x \leqslant y$) and the second component u of the semiflow are of a given type:

this will be a unified notation for three different settings, which we describe in what follows. For future reasons regarding notation, in each case the space X itself is represented as a product. Recall that the semiflow τ is C^1 in x if the linear differential operator with respect to x, $u_x : (0,\infty) \times \Omega \times X \to \mathcal{L}(X,X)$, is well defined and continuous whenever $u(t,\omega,x)$ exists, and it satisfies $\lim_{t\to 0^+} u_x(t,\omega,x)\, y = y$ for every $y \in X$ uniformly for (ω,x) in compact sets.

In the first setting, which will be referred to as *ODEs case*, $X = X_1 \times X_2$ for $X_1 = X_2 = \mathbb{R}$, endowed with the norm $\|x\| = |x_1| + |x_2|$ for $x = (x_1,x_2) \in \mathbb{R}^2$, which is monotone for the strong partial order defined for $x = (x_1,x_2)$ and $y = (y_1,y_2)$ by $x \leqslant y$ if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$; $x < y$ if $x \leqslant y$ and $x \neq y$; and $x \ll y$ if $x_1 < y_1$ and $x_2 < y_2$. Let the function $F = (F_1,F_2) : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ be continuous in its domain and C^1 in $\mathbb{R}^2$. We consider the family of two-dimensional systems of ordinary differential equations (ODEs) given for each element $\omega \in \Omega$ by

$$\begin{cases} x_1'(t) = F_1(\omega\cdot t, x_1(t), x_2(t)), & t \in \mathbb{R}, \\ x_2'(t) = F_2(\omega\cdot t, x_1(t), x_2(t)), & t \in \mathbb{R}, \end{cases}$$

which we write for short as

$$x'(t) = F\big(\omega\cdot t, x(t)\big), \quad t \in \mathbb{R}. \tag{2.2}$$

The standard theory of ODEs permits to define a local continuous skew-product semiflow (actually a flow) (2.1), where $u(t,\omega,x)$ denotes the solution of Eq. (2.2) with initial condition $u(0,\omega,x_0) = x_0$ for $x_0 \in X$, for t in the maximal interval of existence. It is well known that bounded solutions are globally defined, and that the flow is C^1 in x_0, and that $\tau_t(M)$ is relatively compact for any bounded set M for $t > 0$. In particular, bounded orbits are relatively compact.

The second setting is the *delay case*. We consider $X_1 = X_2 = C([-1,0],\mathbb{R})$, the space of real continuous functions in $[-1,0]$ with the sup-norm, and take the product Banach space $X = X_1 \times X_2$ endowed with the norm $\|x\| = \|x_1\| + \|x_2\|$ for $x = (x_1,x_2) \in X$. This norm is monotone for the strong partial order defined in X by the positive cone $X_+ = \{x \in X \mid x(s) \geqslant 0\ \forall s \in [-1,0]\}$, which has nonempty interior $\operatorname{Int} X_+ = \{x \in X \mid x(s) \gg 0\ \forall s \in [-1,0]\}$. In order to define the semiflow, we take a continuous function $F = (F_1,F_2) : \Omega \times \mathbb{R}^4 \to \mathbb{R}^2$ which is C^1 in $\mathbb{R}^4$ and consider the family of two-dimensional systems of finite delay differential equations given for each $\omega \in \Omega$ by

$$\begin{cases} x_1'(t) = F_1(\omega\cdot t, x_1(t), x_2(t), x_1(t-1), x_2(t-1)), & t > 0, \\ x_2'(t) = F_2(\omega\cdot t, x_1(t), x_2(t), x_1(t-1), x_2(t-1)), & t > 0, \end{cases}$$

written for short as

$$x'(t) = F\big(\omega\cdot t, x(t), x(t-1)\big), \quad t > 0. \tag{2.3}$$

The standard theory of delay differential equations (see Hale and Verduyn Lunel [6]) ensures the existence of a unique solution $x(t,\omega,x_0)$ of system (2.3) with initial value $x_0 \in X$ (i.e., $x(s,\omega,x_0) = x_0(s)$ for each $s \in [-1,0]$) for each $\omega \in \Omega$ and $x_0 \in X$, which is locally defined. Therefore, the family (2.3) induces a local continuous skew-product semiflow (2.1), where $u(t,\omega,x_0)(s) = x(t+s,\omega,x_0)$ for $s \in [-1,0]$ and t in the maximal interval of existence. As consequences of Arzelà–Ascoli theorem: if $x(t,\omega,x_0)$ is a bounded solution of Eq. (2.3), for t in its interval of existence, then $u(t,\omega,x_0)$ exists for all $t > 0$; and that $\tau_t(M)$ is relatively compact for any bounded set M for $t > 1$, so that bounded semiorbits for $t \geqslant 1$ are relatively compact in $\Omega \times X$. In addition, the semiflow is C^1 in x_0 (see for instance [6]).

The third and last case is the *parabolic case*. Given a bounded, open and connected subset U of $\mathbb{R}^n$ with a sufficiently smooth boundary ∂U, we consider $X_1 = X_2 = C(\bar{U},\mathbb{R})$, the Banach space of real continuous functions in $\bar{U}$ (the closure of U in $\mathbb{R}^n$) endowed with the sup-norm, and the product Banach space $X = X_1 \times X_2$ with norm $\|x\| = \|x_1\| + \|x_2\|$ for $x = (x_1,x_2) \in X$. This norm is also monotone for the strong partial order defined on X by the positive cone $X_+ = \{x \in X \mid x(v) \geqslant 0\ \forall v \in \bar{U}\}$, whose (nonempty) interior is $\operatorname{Int} X_+ = \{x \in X \mid x(v) \gg 0 \ \forall v \in \bar{U}\}$. In this case, the semiflow will be associated to the family of two-dimensional systems of parabolic partial differential equations with Neumann boundary conditions given for each $\omega \in \Omega$ by

$$\begin{cases} \partial_t x_1(t,v) = d_1 \Delta x_1(t,v) + F_1(\omega\cdot t, v, x_1(t,v), x_2(t,v)), & t>0,\ v \in U, \\ \partial_t x_2(t,v) = d_2 \Delta x_2(t,v) + F_2(\omega\cdot t, v, x_1(t,v), x_2(t,v)), & t>0,\ v \in U, \\ (\partial x_i/\partial n)(t,v) = 0, & t>0,\ v\in \partial U,\ i=1,2, \end{cases}$$

for $F = (F_1, F_2) : \Omega \times \bar{U} \times \mathbb{R}^2 \to \mathbb{R}^2$, where Δ represents the Laplacian operator on $\mathbb{R}^n$, d_1, d_2 are positive constants, and $\partial/\partial n$ denotes the exterior normal derivative at the boundary. We write these equations for short as

$$\begin{cases} \partial_t x(t,v) = D\Delta x(t,v) + F(\omega\cdot t, v, x(t,v)), & t>0,\ v\in U, \\ (\partial x/\partial n)(t,v) = 0, & t>0,\ v \in \partial U, \end{cases} \tag{2.4}$$

where D is the diagonal matrix with entries d_1 and d_2. The function F is supposed to be continuous in its domain, C^2 in v and x, and such that for each $\omega \in \Omega$, $v \in \bar{U}$ and $x \in \mathbb{R}^2$, the functions $\mathbb{R} \to \mathbb{R}^2$, $t \mapsto F(\omega\cdot t, v, x_1, x_2)$ and $\mathbb{R} \to \mathbb{R}^2$, $t \mapsto (\partial F/\partial x_i)(\omega\cdot t, v, x_1, x_2)$ for $i = 1, 2$ are Lipschitz. These conditions ensure the existence of a unique local solution $x_{\omega,x_0}(t,v)$ of (2.4) with initial condition $x_{\omega,x_0}(0,v) = x_0(v)$ for $v \in \bar{U}$ for each $\omega \in \Omega$ and $x_0 \in X$. This solution is *classical*: its partial derivatives exist, are continuous and satisfy the corresponding equations. See Friedman [5], Lunardi [18] and Smith [30] for further details. Let us represent by $u(t,\omega,x_0) \in X$ the function given by $u(t,\omega,x_0)(v) = x_{\omega,x_0}(t,v)$ for $v \in \bar{U}$ and t in the maximal interval of existence. As in the previous cases, bounded solutions are globally defined and $\tau_t(M)$ is relatively compact for any bounded set M for $t > 0$ (see Travis and Webb [33]), which implies that bounded semiorbits for $t \geqslant \delta > 0$ are relatively compact. In addition, the semiflow C^1 in x_0 (see [18] and Henry [7]).

Often, these families of systems come from a single one: a nonautonomous two-dimensional differential system of ordinary, finite delay, or parabolic type, for which the coefficient function $f = (f_1, f_2)$ satisfies suitable conditions of *admissibility* and *recurrence*. These conditions guarantee that the corresponding *hull* Ω is a compact metric space, that the time-translation flow on it is continuous and minimal, and that all the functions in Ω inherit the same properties regarding continuity and Lipschitz or C^1 character as the initial one. The reader can find in [27] a detailed description of the admissibility conditions.

2.2. Reformulation of the hypotheses

As next step, we will describe conditions on the families of equations guaranteeing the monotonicity and concavity of the induced semiflow τ. Recall that a monotone and concave semiflow given on the fiber by u satisfies

$$\begin{aligned} &u(t,\omega,z) \geqslant \lambda u(t,\omega,y) + (1-\lambda)u(t,\omega,x) \quad \text{for } \omega\in\Omega, \lambda\in[0,1] \quad \text{and} \\ &z \geqslant \lambda y + (1-\lambda)x \quad \text{with } x \leqslant y, \\ &u(t,\omega,z) \leqslant \lambda u(t,\omega,y) + (1-\lambda)u(t,\omega,x) \quad \text{for } \omega\in\Omega, \lambda\notin(0,1) \quad \text{and} \\ &z \leqslant \lambda y + (1-\lambda)x \quad \text{with } x \leqslant y \end{aligned} \tag{2.5}$$

for those values of $t \geqslant 0$ for which all the terms are defined. The definitions of monotonicity and concavity are respectively given by the previous inequalities for $\lambda = 1$ and $z = \lambda y + (1-\lambda)x$.

We begin by describing the cooperativity conditions on F, frequently referred to as *quasi-monotonicity* conditions in the delay and parabolic cases.

Definition 2.1. (i) Family (2.2) is *cooperative* if $\partial F_1/\partial x_2 \geqslant 0$ and $\partial F_2/\partial x_1 \geqslant 0$ on $\Omega \times \mathbb{R}^2$.

(ii) Family (2.3) is *cooperative* if $\partial F_1/\partial x_i \geqslant 0$ for $i = 2, 3, 4$ and $\partial F_2/\partial x_i \geqslant 0$ for $i = 1, 3, 4$ on $\Omega \times \mathbb{R}^4$.

(iii) Family (2.4) is *cooperative* if $\partial F_1/\partial x_2 \geqslant 0$ and $\partial F_2/\partial x_1 \geqslant 0$ on $\Omega \times \bar{U} \times \mathbb{R}^2$.

In the case that the family we consider comes from a single system, the previous conditions hold if and only if the corresponding partial derivatives of the coefficient functions f_1 and f_2 are nonnegative. Let us now describe the concavity conditions.

Definition 2.2. (i) Family (2.2) is *concave* if $F(\omega, \lambda y + (1-\lambda)x) \geqslant \lambda F(\omega, y) + (1-\lambda)F(\omega, x)$ for any $\omega \in \Omega$, $\lambda \in [0,1]$, and $x, y \in \mathbb{R}^2$ with $x \leqslant y$.

(ii) Family (2.3) is *concave* if $F(\omega, \lambda y + (1-\lambda)x) \geqslant \lambda F(\omega, y) + (1-\lambda)F(\omega, x)$ for any $\omega \in \Omega$, $\lambda \in [0,1]$, and $x, y \in \mathbb{R}^4$ with $x \leqslant y$.

(iii) Family (2.4) is *concave* if $F(\omega, v, \lambda y + (1-\lambda)x) \geqslant \lambda F(\omega, v, y) + (1-\lambda)F(\omega, v, x)$ for any $\omega \in \Omega$, $v \in \bar{U}$, $\lambda \in [0,1]$, and $x, y \in \mathbb{R}^2$ with $x \leqslant y$.

Again, all the elements of the hull of an admissible and recurrent initial function f are concave if f is. In the rest of the paper, we will work with the following hypotheses on the family (2.2), (2.3) or (2.4):

(H1) The function F satisfies the regularity conditions previously assumed.
(H2) The family is cooperative and concave.

The semiflow inherits regularity, monotonicity and concavity from the function F:

Proposition 2.3. *Assume that the family* (2.2), (2.3) *or* (2.4) *satisfies hypotheses* (H1) *and* (H2). *Then, the induced skew-product semiflow on the corresponding space $\Omega \times X$ is C^1 in x, monotone and concave.*

Proof. The properties of monotonicity and concavity follow from standard arguments of comparison of solutions. See for instance Smith [30] for ODEs and delay equations and Fife and Tang [4] for PDEs, as well as the arguments of Proposition 2.3 in [27]. □

This result and the previously mentioned properties of the semiorbits ensure that the semiflows we consider satisfy hypotheses (h1) and (h2) in [28]: i.e., they are C^1 in x, monotone and concave, and any bounded semiorbit is globally defined and relatively compact. In order to complete the hypotheses required to start with the analysis, we substitute hypothesis (h3) in that paper (see below) by a slightly different one. Recall that a *subequilibrium* for τ is a map $a : \Omega \to X$ such that $u(t, \omega, a(\omega))$ is defined for any $t \geqslant 0$ and $\omega \in \Omega$ with $a(\omega \cdot t) \leqslant u(t, \omega, a(\omega))$, and that is it *semicontinuous* if $\{(\omega, a(\omega)) \in \Omega \times X \mid \omega \in \Omega\}$ and $\{(\omega, x) \in \Omega \times X \mid x \geqslant a(\omega)\}$ are respectively relatively compact and closed in $\Omega \times X$. The notion of *semicontinuous superequilibrium* is analogous. And a is an *equilibrium* if $a(\omega \cdot t) = u(t, \omega, a(\omega))$, and it is *semicontinuous* if it is so as sub or superequilibrium. The mentioned hypothesis is

(H3) the semiflow τ admits a semicontinuous subequilibrium a, and a bounded semiorbit $\{(\omega \cdot t, u(t, \omega, x)) \mid t \geqslant 0\}$ such that, for an $e \gg 0$, $u(t, \omega, x) \geqslant a(\omega \cdot t) + e$ for every $t \geqslant 0$. In addition, in the delay case, $a(\omega)(s) = a(\omega \cdot s)(0)$ for every $\omega \in \Omega$ and $s \in [-1, 0]$.

It is clear that (H3) is stronger than (h3) in [28]: i.e., there exist a semicontinuous subequilibrium a, a vector $e \gg 0$, and a minimal set K such that $a(\omega) + e \leqslant y$ for any $(\omega, y) \in K$, situation that we represent as $a \ll K$ and which implies that the semiflow restricted to $C_a = \{(\omega, x) \mid x \geqslant a(\omega)\}$ is globally defined (see [28, Propositions 3.4 and 3.6]). In fact both conditions are almost equivalent: there is just an extra condition in (H3) for the delay case, always satisfied if a is an equilibrium, which is required to guarantee the next componentwise separation property, fundamental in what follows.

Proposition 2.4.

(i) *Assume that the ODEs or parabolic family* (2.2) *or* (2.4) *satisfies* (H1) *and* (H2). *Let* $d=(d_1,d_2)$ *be a semicontinuous subequilibrium. If the points* (ω,x) *and* (ω,y) *satisfy* $x\geqslant d(\omega)$ *or* $x\leqslant d(\omega)$ *and, for a* $\lambda\in[0,1]$, $y\geqslant \lambda x+(1-\lambda)d(\omega)$ *and* $y_i>\lambda x_i+(1-\lambda)d_i(\omega)$ *for* $i=1$ *or* $i=2$, *then* $u_i(t,\omega,y)\gg \lambda u_i(t,\omega,x)+(1-\lambda)d_i(\omega\cdot t)$ *for any* $t>0$ *such that all the terms involved are defined.*

(ii) *Assume that the delay family* (2.3) *satisfies* (H1) *and* (H2). *Let* $d=(d_1,d_2)$ *be a semicontinuous subequilibrium such that* $d(\omega)(s)=d(\omega\cdot s)(0)$ *for every* $\omega\in\Omega$ *and* $s\in[-1,0]$. *If the points* (ω,x) *and* (ω,y) *admit backward orbits* $\{(\omega\cdot s,x^s)\mid s\in[-1,0]\}$ *and* $\{(\omega\cdot s,y^s)\mid s\in[-1,0]\}$ *and satisfy* $x^s\geqslant d(\omega\cdot s)$ *or* $x^s\leqslant d(\omega\cdot s)$ *for every* $s\in[-1,0]$, *and for a* $\lambda\in[0,1]$, $y^s\geqslant\lambda x^s+(1-\lambda)d(\omega\cdot s)$ *for every* $s\in[-1,0]$ *and* $y_i>\lambda x_i+(1-\lambda)d_i(\omega)$ *for* $i=1$ *or* $i=2$, *then* $u_i(t,\omega,y)\gg\lambda u_i(t,\omega,x)+(1-\lambda)d_i(\omega\cdot t)$ *for any* $t\geqslant 1$ *such that all the terms involved are defined.*

Proof. The proof is quite similar to the one of Proposition 2.4 in [27], but we include it for the reader's convenience. We jump the ODEs case, which is the simplest one.

For the delay case, we represent $x^s=u(s,\omega,x)$ and $y^s=u(s,\omega,y)$ for $s\in[-1,0]$, and look for $\tilde s\in[-1,0]$ such that $y_i(\tilde s)>\lambda x_i(\tilde s)+(1-\lambda)d_i(\omega)(\tilde s)$. Then, taking as initial data $u(\tilde s,\omega,y)\geqslant\lambda u(\tilde s,\omega,x)+(1-\lambda)d(\omega\cdot\tilde s)$, which satisfy

$$u_i(\tilde s,\omega,y)(0)=y_i(\tilde s)>\lambda\, x_i(\tilde s)+(1-\lambda)d_i(\omega)(\tilde s)=\big(\lambda u_i(\tilde s,\omega,x)+(1-\lambda)d_i(\omega\cdot\tilde s)\big)(0),$$

we linearize the problem in order to deduce from Lemma 5.1.3 in [30] for linear delay equations, from (2.5) and from the definition of subequilibrium that

$$\begin{aligned}x_i(t+\tilde s,\omega,y)&=x_i\big(t,\omega\cdot\tilde s,y^{\tilde s}\big)>x_i\big(t,\omega\cdot\tilde s,\lambda\, x^{\tilde s}+(1-\lambda)d(\omega\cdot\tilde s)\big)\\&\geqslant\lambda x_i\big(t,\omega\cdot\tilde s,x^{\tilde s}\big)+(1-\lambda)x_i\big(t,\omega\cdot\tilde s,d(\omega\cdot\tilde s)\big)\\&\geqslant\lambda x_i(t+\tilde s,\omega,x)+(1-\lambda)d_i\big(\omega\cdot(t+\tilde s)\big)(0)\end{aligned}$$

for any $t\geqslant 0$. Consequently, if $t\geqslant 1$ and $s\in[-1,0]$, since $t+s-\tilde s\geqslant 0$,

$$\begin{aligned}u_i(t,\omega,y)(s)&=x_i(t+s,\omega,y)>\lambda\, x_i(t+s,\omega,x)+(1-\lambda)d_i\big(\omega\cdot(t+s)\big)(0)\\&=\lambda u_i(t,\omega,x)(s)+(1-\lambda)d_i(\omega{\cdot}t)(s),\end{aligned}$$

which proves the assertion.

In the parabolic case, the function $z_i(t,v)=u_i(t,\omega,y)(v)-u_i(t,\omega,\lambda x+(1-\lambda)d(\omega))(v)$ is non-negative for any $t\geqslant 0$, as deduced from (2.5). By hypothesis there is $v_0\in\bar U$ such that $z_i(0,v_0)=y_i(v_0)-\lambda x_i(v_0)-(1-\lambda)d_i(\omega)(v_0)>0$. We apply to z_i on $[0,t+1]\times\bar U$ the minimum principle for parabolic PDEs, by linearizing the problem (see for instance [4] and Section 7.2 in [30]), in order to conclude that $z_i(t,v)>0$ for any $t>0$ and $v\in\bar U$. The result then follows from (2.5). □

2.3. C^1 subequilibria and differential inequations

Let us now analyze the close relation between the concept of subequilibrium and the better known concept of lower solution, whose existence is, in general, easier to check.

Definition 2.5. (i) A map $a:\Omega\to\mathbb R^m$ is C^1 *along the base orbits* if, for each $\omega\in\Omega$, the map $\mathbb R\to\mathbb R^m$, $t\mapsto a(\omega\cdot t)$ is continuously differentiable, in which case we represent $a'(\omega)=(d/dt)a(\omega\cdot t)|_{t=0}$. The map a is C^1 *in* Ω if, in addition, a and a' are continuous in Ω.

(ii) A map $a:\Omega\to C([-1,0],\mathbb R^m)$ is C^1 *along the base orbits* if the map $\bar a:\Omega\to\mathbb R^m$, $\omega\mapsto a(\omega)(0)$ is, and it is C^1 *in* Ω if, in addition, a and $\bar a'$ are continuous in Ω.

(iii) A map $a : \Omega \to C(\bar{U}, \mathbb{R}^m)$ is C^1 *along the base orbits* if, for each $v \in \bar{U}$, the map $\bar{a}(\cdot, v) : \Omega \to \mathbb{R}^m$, $\omega \mapsto a(\omega)(v)$ is. In this case we represent $\bar{a}'(\omega, v) = \partial_t \bar{a}(\omega \cdot t, v)|_{t=0}$. The map a is C^1 *in* Ω if, in addition, a is continuous in Ω and $\bar{a}'$ is continuous in $\Omega \times \bar{U}$.

From now on, the notations $\bar{a}(\omega) = a(\omega)(0)$ and $\bar{a}'(\omega) = (d/dt)\bar{a}(\omega \cdot t)|_{t=0}$ for $a : \Omega \to C([-1, 0], \mathbb{R}^2)$, and $\bar{a}(\omega, v) = a(\omega)(v)$ and $\bar{a}'(\omega, v) = (d/dt)\bar{a}(\omega \cdot t, v)|_{t=0}$ for $a : \Omega \to C(\bar{U}, \mathbb{R}^2)$ will often be used. In the delay case, a function $\bar{a} : \Omega \to \mathbb{R}^2$ allows us to define $a : \Omega \to C([-1, 0], \mathbb{R}^2)$ by $a(\omega)(s) = \bar{a}(\omega \cdot s)$. Note that $a(\omega \cdot s)(0) = a(\omega)(s)$ for every $\omega \in \Omega$ and $s \in [-1, 0]$.

The next results, formulated for simplicity for two-dimensional systems, are also valid in the m-dimensional case for $m \geqslant 1$. And reversing the inequalities, the analogous results for superequilibria are obtained.

Proposition 2.6.

(i) *Let $a : \Omega \to \mathbb{R}^2$ be C^1 along the base orbits. Then a is a subequilibrium for the family* (2.2) *of ordinary differential equations if and only if, for every $\omega \in \Omega$, $a'(\omega) \leqslant F(\omega, a(\omega))$ and $u(t, \omega, a(\omega))$ is defined for every $t \geqslant 0$.*

(ii) *Let $a : \Omega \to C([-1, 0], \mathbb{R}^2)$ be C^1 along the base orbits and satisfy $a(\omega)(s) = a(\omega \cdot s)(0)$ for every $\omega \in \Omega$ and $s \in [-1, 0]$. Then a is a subequilibrium for the family* (2.3) *of delay equations if and only if, for every $\omega \in \Omega$, $\bar{a}'(\omega) \leqslant F(\omega, \bar{a}(\omega), \bar{a}(\omega \cdot (-1)))$ and $u(t, \omega, a(\omega))$ is defined for every $t \geqslant 0$.*

(iii) *Let $a : \Omega \to C(\bar{U}, \mathbb{R}^2)$ be C^1 along the base orbits and such that, for every $\omega \in \Omega$, the map $\mathbb{R} \times \bar{U} \to \mathbb{R}^2$, $(t, v) \mapsto \bar{a}(\omega \cdot t, v)$ is continuously differentiable on its domain, twice continuously differentiable in $v \in U$, and with $(\partial\bar{a}/\partial n)(\omega, v) = 0$ for $v \in \partial U$. Then a is a subequilibrium for the family* (2.4) *of parabolic equations if and only if, for any $\omega \in \Omega$ and $v \in \bar{U}$, $\bar{a}'(\omega, v) \leqslant D\Delta\bar{a}(\omega, v) + F(\omega, v, \bar{a}(\omega, v))$ and $u(t, \omega, a(\omega))$ is defined for every $t \geqslant 0$.*

Proof. The if part of each statement is proved as Proposition 4.4 in [21] and Lemma 2.11 in [27]. In the ODEs case, the converse statement is deduced from the fact that, for $i = 1, 2$, the scalar function $u_i(t, \omega, a(\omega)) - a_i(\omega \cdot t)$ of t vanishes at $t = 0$ and is positive for $t > 0$, so that its derivative at $t = 0$ must be nonnegative. The same happens in the delay case and the parabolic cases, working now with the right-derivative at $t = 0$. $\square$

Remark 2.7. The same arguments prove that, in the case that only one component a_i of the subequilibrium a is C^1 along the base orbits, with $a(\omega)(s) = a(\omega \cdot s)(0)$ in the delay case, this component satisfies the corresponding differential inequality.

It is also possible to establish a relation between strong subequilibria and differential inequations which are strict at some point with additional regularity properties. See [21] and [27] for further details.

2.4. Copies of the base and marginal semiflows

We complete this section by describing the so-called *marginal* semiflows associated to a copy of the base $K = \{c\} = \{c_1, c_2\}$; or, in other words, to a continuous equilibrium $c : \Omega \to X$.

Remark 2.8. Let $K = \{c_1, c_2\}$ be a copy of the base. Then, for $i = 1, 2$, in the ODEs case,

$$c_i'(\omega) = F_i\big(\omega, c_1(\omega), c_2(\omega)\big), \quad \omega \in \Omega;$$

in the delay case, $c_i(\omega)(s) = \bar{c}_i(\omega \cdot s)$ for any $\omega \in \Omega$ and $s \in [-1, 0]$, and

$$\bar{c}_i'(\omega) = F_i\big(\omega, \bar{c}_1(\omega), \bar{c}_2(\omega), \bar{c}_1\big(\omega \cdot (-1)\big), \bar{c}_2\big(\omega \cdot (-1)\big)\big), \quad \omega \in \Omega;$$

and, in the parabolic case, for each $\omega \in \Omega$ the map $(t, v) \mapsto \bar{c}_i(\omega \cdot t, v)$ is continuously differentiable in its domain and twice continuously differentiable in v for $t = 0$, and

$$\begin{cases} \bar{c}_i'(\omega, v) = d_i \Delta \bar{c}_i(\omega, v) + F_i(\omega, v, \bar{c}_1(\omega, v), \bar{c}_2(\omega, v)), & \omega \in \Omega,\ v \in U, \\ (\partial \bar{c}_i/\partial n)(\omega, v) = 0, & \omega \in \Omega,\ v \in \partial U. \end{cases}$$

In addition, in the parabolic case, the functions $\bar{c}_i$ and $\Delta \bar{c}_i$ are continuous, as deduced from the results of [18]. The conclusion in that, in the three cases, the map (c_1, c_2) is a C^1 equilibrium.

To begin with the construction of the marginal semiflows, we fix the second component c_2 of K, and consider the family of scalar ODEs

$$x_1'(t) = F_1\big(\omega \cdot t, x_1(t), c_2(\omega \cdot t)\big), \quad t \in \mathbb{R};$$

or of scalar delay differential equations:

$$x_1'(t) = F_1\big(\omega \cdot t, x_1(t), \bar{c}_2(\omega \cdot t), x_1(t-1), \bar{c}_2\big(\omega \cdot (t-1)\big)\big), \quad t > 0,$$

or of scalar parabolic PDEs:

$$\begin{cases} \partial_t x_1(t, v) = d_1 \Delta x_1(t, v) + F_1(\omega \cdot t, v, x_1(t, v), \bar{c}_2(\omega \cdot t, v)), & t > 0,\ v \in U, \\ (\partial x_1/\partial n)(t, v) = 0, & t > 0,\ v \in \partial U. \end{cases}$$

Following the indications given in Section 2.4 in [27], it is easy to check that each one of these three families of equations satisfies conditions (H1) and (H2), so that its solutions define a continuous semiflow

$$\tau_{1,c_2} : \mathbb{R}_+ \times \Omega \times X_1 \to \Omega \times X_1, \qquad (t, \omega, x_1) \mapsto \big(\omega{\cdot}t, u_1^{c_2}(t, \omega, x_1)\big), \tag{2.6}$$

which satisfies (h1) and (h2) in [28]. Fixing the first component c_1 of K, we obtain a second monotone and concave semiflow

$$\tau_{2,c_1} : \mathbb{R}_+ \times \Omega \times X_2 \to \Omega \times X_2, \qquad (t, \omega, x_2) \mapsto \big(\omega{\cdot}t, u_2^{c_1}(t, \omega, x_2)\big), \tag{2.7}$$

with the analogous properties. The main advantage of working with these marginal semiflows, what we will often do in what follows, is that they satisfy the five hypotheses (h1)–(h4) and (h4$^+$) in [28]. Since hypotheses (h1)–(h3) have already been recalled, we repeat here the last two ones:

- (h4) There exist $\tilde{\omega} \in \Omega$ and $\tilde{t} > 0$, with a continuous at $\tilde{\omega}{\cdot}\tilde{t}$, such that if the points $(\tilde{\omega}, x)$ and $(\tilde{\omega}, y)$ respectively belong to minimal sets K_1 and K_2 with $K_1 \geqslant a$, and they have backward orbits $\{(\tilde{\omega} \cdot s, x_s) \mid s \leqslant 0\}$ and $\{(\tilde{\omega} \cdot s, y_s) \mid s \leqslant 0\}$ such that for a $\lambda \in [0, 1]$ it is $y_s \geqslant \lambda x_s + (1-\lambda) a(\tilde{\omega} \cdot s)$ for any $s \leqslant 0$, and $u(t, \tilde{\omega}, y) > \lambda u(t, \tilde{\omega}, x) + (1-\lambda) a(\tilde{\omega} \cdot t)$ for certain $t \geqslant 0$, then $u(t+\tilde{t}, \tilde{\omega}, y) \gg \lambda u(t+\tilde{t}, \tilde{\omega}, x) + (1-\lambda) a(\tilde{\omega} \cdot (t+\tilde{t}))$.
- (h4$^+$) Hypothesis (h4) holds when a is replaced by any continuous equilibrium d and the point $(\tilde{\omega}, x)$ belongs to a minimal set K_1 with $K_1 \leqslant d$ or $K_1 \geqslant d$.

Proposition 2.9. *Assume that the family* (2.2), (2.3) *or* (2.4) *satisfies* (H1)–(H3). *If* $K = \{c_1, c_2\} \gg a$ *is a copy of the base, then the marginal semiflows* τ_{i,c_j} *for* $i \neq j$ *satisfy hypotheses* (h1)–(h3) *in* [28]. *In addition, the components of any subequilibrium below* c (*resp. superequilibrium above* c) *for* τ *provide subequilibria* (*resp. superequilibria*) *for these semiflows. Finally, they also satisfy the next condition*:

There exists $\tilde{t} > 0$ such that, if d is a semicontinuous τ_{i,c_j}-subequilibrium, the points (ω, x) and (ω, y) respectively belong to τ_{i,c_j}-minimal sets K_1 and K_2 with $K_1 \geqslant d$ or $K_1 \leqslant d$, and they have backward orbits $\{(\omega \cdot s, x_s) \mid s \leqslant 0\}$ and $\{(\omega \cdot s, y_s) \mid s \leqslant 0\}$ such that for a $\lambda \in [0,1]$ it is $y_s \geqslant \lambda x_s + (1-\lambda)d(\omega \cdot s)$ for any $s \leqslant 0$, and $u_i^{c_j}(t, \omega, y) > \lambda u_i^{c_j}(t, \omega, x) + (1-\lambda)d(\omega \cdot t)$ for certain $t \geqslant 0$, then $u_i^{c_j}(t+\tilde{t}, \omega, y) \gg \lambda u_i^{c_j}(t+\tilde{t}, \omega, x) + (1-\lambda)d(\omega \cdot (t+\tilde{t}))$.

In particular, the marginal semiflows also satisfy hypotheses (h4) *and* (h4$^+$) *in* [28].

Proof. We have already mentioned that conditions (H1)–(H2) for τ guarantee that the scalar semiflows τ_{1,c_2} and τ_{2,c_1} satisfy (h1) and (h2) in [28]. It follows from the equations in Remark 2.8 that $\{c_i\}$ is a minimal set for the semiflow τ_{i,c_j}, with $j \neq i$. By applying again standard arguments of comparison of solutions, as in the proof of Proposition 2.4, we also check that a_i as well as the ith component of any subequilibrium below c (resp. superequilibrium above c) are semicontinuous subequilibria (resp. superequilibria) for τ_{i,c_j}. So that τ_{i,c_j} also satisfies (h3) in [28] if (H3) holds for τ. The last property is proved by arguing again as in the proof of Proposition 2.4. □

Note that $\{c_1, c_2\}$ is a τ-minimal set if and only if $\{c_1\}$ is a τ_{1,c_2}-minimal set and $\{c_2\}$ is a τ_{2,c_1}-minimal set. This property will be repeatedly used. We finally point out that in the case of the existence of a minimal set K whose elements can be written as $(\omega, x_1, c_2(\omega))$ for a continuous map c_2, the previous equations also define a semiflow, which we denote by τ_{1,c_2}, and that $K_1 = \{(\omega, x_1) \mid (\omega, x_1, c_2(\omega)) \in K\}$ is a τ_{1,c_2}-minimal set.

2.5. Some previous basic results

As already mentioned, the results of this paper strongly rely on those in [28], mainly on Theorems 3.8 and 3.13 in that paper. We include a summary of the most significative properties there stated.

Theorem 3.8 establishes two different possibilities for the global dynamics above a under hypotheses (h1)–(h3). In Case A1, K is the unique minimal set strongly above a, and it is a copy of the base to which any semiorbit which is eventually strongly above a approaches (exponentially, in fact). On the contrary, in Case A2: there are infinitely many minimal sets strongly above a; given any of these sets there is another one strictly below it; the union of all of them compose a connected subset of the phase space, which can be bounded or unbounded; and its boundedness is equivalent to the existence of a top minimal set K^+, which is a new copy of the base attracting asymptotically any semiorbit eventually above it.

Theorem 3.13 gives a much more detailed description of the global dynamics (not only above a) in Case A2 under the additional hypotheses (h4) and (h4$^+$). Among other properties, it shows the existence of a continuous equilibrium $\tilde{a} \geqslant a$ agreeing with a at its continuity points, a continuous equilibrium $c^1 \gg \tilde{a}$, and a closed (maybe unbounded) interval $J \subset R$ containing $[0,1]$ such that M is a minimal set of the semiflow if and only if $M = \{c^\lambda\}$ for $c^\lambda = \lambda c^1 + (1-\lambda)\tilde{a}$ for some $\lambda \in J$. Of course, $\sup J = \lambda_+ < \infty$ if and only if c^{λ_+} provides the top minimal set. And finally, if $\lambda_- = \inf J > -\infty$ and $x \ll c^{\lambda_-}(\omega)$, then the semiorbit of (ω, x) is unbounded.

3. The dynamics above a semicontinuous subequilibrium

Throughout the rest of the paper we will always assume that the family (2.2), (2.3) or (2.4) satisfies hypotheses (H1)–(H3), which as seen in the previous section allows us to apply all the results established in the paper [28] under its assumptions (h1)–(h3).

In particular, Theorem 3.8 in [28] gives a precise description of the dynamics on the area strongly above the subequilibrium a when "Case A1" holds; i.e., when there exists a unique minimal set strongly above the subequilibrium a. As mentioned there, Example 3.15 below shows the optimality of the description with a sample of this situation for a non-continuous subequilibrium a.

Excepting in that example, we assume in this section that the dynamics fits “Case A2” of the theorem: there exist infinitely many minimal sets strongly above a. Our goal is to describe all the elements of the set

$$\mathcal{M} = \{K \subset \Omega \times X \mid K \text{ is minimal}\},$$

which contains the infinitely many elements of

$$\mathcal{M}_a = \{K \subset \Omega \times X \mid K \text{ is minimal and } a \ll K\}.$$

This information must be added to the one provided by Proposition 3.6, Theorem 3.8 and Remark 3.9.1 in [28] in order to get a global idea of the dynamics on the area of $\Omega \times X$ strongly above the initial subequilibrium a.

Let us begin by checking that any element of $\mathcal{M}_a$ is a copy of the base.

Theorem 3.1. *Assume that the family* (2.2), (2.3) *or* (2.4) *satisfies* (H1) *and* (H2). *If it also satisfies* (H3) *and* $(\omega, x) \in \Omega \times X$ *satisfies* $x \gg a(\omega)$, *its omega-limit set is a copy of the base strongly above a. And if there exist two strongly ordered minimal sets* $M_1 \ll M_2$, *then* M_2 *is also a copy of the base.*

Proof. In the first situation, Proposition 3.6 in [28] shows that any semiorbit starting strongly above a is globally defined and uniformly stable and that its omega-limit set is minimal and strongly above a. The statement follows from results of Jiang and Zhao [10] and Novo et al. [23]. (See also the proof of Theorem 2.5 in [27].) The same arguments work in the second case. □

To obtain the global description of $\mathcal{M}$, we adapt to the concave case the ideas of [27], applied there to the sublinear case for which the null map is a continuous subequilibrium. As an auxiliary tool, we fix a *reference* minimal set $K^1 = \{c^1\} = \{c_1^1, c_2^1\} \gg a$. We fix $\tilde{\omega} \in \Omega$ and define the maps $c^\lambda = (c_1^\lambda, c_2^\lambda)$ for $\lambda \in [0,1]$ and the minimal sets $K^\lambda \gg a$ for $\lambda \in (0,1]$ by

$$c^\lambda = \lambda c^1 + (1-\lambda)a \quad \text{and} \quad K^\lambda = \mathcal{O}\big(\tilde{\omega}, c^\lambda(\tilde{\omega})\big). \tag{3.1}$$

It is immediate to deduce from (2.5) that c^λ is a semicontinuous subequilibrium and $K^\lambda \geqslant c^\lambda$. According to Proposition 3.7 and Theorem 3.8 in [28], since the dynamics is supposed to fit Case A2, the infinitely many sets $(K^\lambda)_{\lambda \in (0,1]}$ are all different elements of the set

$$\mathcal{M}_{a,K^1} = \big\{K \in \mathcal{M} \,\big|\, a \ll K \leqslant K^1\big\} \subseteq \mathcal{M}_a,$$

and a priori nothing precludes the existence of more elements. Our next purpose is to describe the *labeling of elements of* $\mathcal{M}_{a,K^1}$ *with respect to* K^1. We fix $\omega \in \Omega$. We say that $K = \{c\} \in \mathcal{M}_{a,K^1}$ is *labeled by* λ (w.r.t. K^1), and write it as $l_{K^1}(K) = \lambda$, when λ is the largest real number such that $c(\omega) \geqslant c^\lambda(\omega)$. The minimality of Ω, the continuity of the equilibrium c and the semicontinuity of the subequilibrium a ensure that λ is independent of the choice of ω.

Proposition 3.2. *Assume that the family* (2.2), (2.3) *or* (2.4) *satisfies* (H1)–(H3), *and that the dynamics above a fits Case* A2 *of Theorem* 3.8 *in* [28]. *Then,*

(i) $l_{K^1}(K) > 0$ *for any* $K \in \mathcal{M}_{a,K^1}$, *and* $l_{K^1}(K) = 1$ *if and only if* $K = K^1$.

(ii) *If* $\lambda = l_{K^1}(K)$ *for* $K = \{c_1, c_2\} \in \mathcal{M}_{a,K^1}$, *one of these three cases holds*: $K = \{c_1^\lambda, c_2\}$ *with* $c_2 \gg c_2^\lambda$, *in which case* a_1 *is* C^1; $K = \{c_1, c_2^\lambda\}$ *with* $c_1 \gg c_1^\lambda$, *in which case* a_2 *is* C^1; *or* $K = \{\tilde{c}_1^\lambda, \tilde{c}_2^\lambda\}$, *with* $\tilde{c}_i^\lambda \geqslant c_i^\lambda$ *and* $\tilde{c}_i^\lambda|_R = c_i^\lambda|_R$ *for a residual subset* R *of* Ω *and* $i = 1, 2$, *in which case there exists a* C^1 *equilibrium* $\tilde{a} \geqslant a$ *which agrees with a in* R *and such that* $\tilde{c}_i^\lambda = \lambda c_i^1 + (1-\lambda)\tilde{a}_i$ *for* $i = 1, 2$.

(iii) $l_{K^1}(K^\lambda) = \lambda$ *for every* $\lambda \in (0,1]$.

(iv) $l_{K^1}(K) \geqslant \lambda$ *for* $K = \{c\}$ *if and only if* $c \geqslant c^\lambda$; *i.e., if and only if* $K \geqslant K^\lambda$. *In particular, if* $K = \{\tilde{c}_1^\lambda, \tilde{c}_2^\lambda\}$ *as described in* (ii), *then* $K = K^\lambda$.

Proof. (i) These properties follow immediately from the definition of the label.

(ii) Let R be the set of continuity points of a, and hence of c^λ for any $\lambda \in [0,1)$, which is a residual subset of Ω (see Proposition 3.2 in [28]). Assume that $c_1(\omega_0) > c_1^\lambda(\omega_0)$ for an $\omega_0 \in R$. To check that $c_1 \gg c_1^\lambda$, we fix any $\omega \in \Omega$. If $c_1(\omega\cdot(-t)) = c_1^\lambda(\omega\cdot(-t))$ for any $t > 1$, the minimality of Ω ensures that $c_1(\omega_0) = c_1^\lambda(\omega_0)$. Since this is not the case, there is $t_1 \geqslant 1$ with $c_1(\omega\cdot(-t_1)) > c_1^\lambda(\omega\cdot(-t_1))$. We apply Proposition 2.4 to the subequilibrium a, and the points $(\omega\cdot(-t_1), x_0) = (\omega\cdot(-t_1), c^1(\omega\cdot(-t_1)))$ and $(\omega\cdot(-t_1), y_0) = (\omega\cdot(-t_1), c(\omega\cdot(-t_1)))$ in order to conclude that $c_1(\omega) \gg c_1^\lambda(\omega)$, as asserted. Now assume by contradiction that $c_2(\omega) > c_2^\lambda(\omega)$ for an $\omega \in \Omega$ and apply Proposition 2.4 to conclude that $c_2(\omega\cdot 1) \gg c_2^\lambda(\omega\cdot 1)$, which contradicts the definition of the label λ. Consequently, $c_2 = c_2^\lambda$ and a_2 is C^1.

The remaining case to be analyzed is $c_i(\omega) = c_i^\lambda(\omega)$ at every point $\omega \in R$ for $i = 1,2$, in which case we define $\tilde{c}_i^\lambda = c_i$ for $i = 1,2$. By the definition of the label, $\tilde{c}_i^\lambda \geqslant c_i^\lambda$. We define the C^1 map $\tilde{a} = (1/(1-\lambda))\tilde{c}^\lambda - (\lambda/(1-\lambda))c^1 \geqslant a$. We fix $\omega \in \Omega$ and $t > 0$, and write $\omega = \lim_{n\to\infty}\omega_n$ for a suitable sequence of points $(\omega_n) \subset R$ such that $(\omega_n \cdot t) \subset R$, possible since the map $\omega \mapsto a(\omega\cdot t)$ has also a residual set of continuity points. Then $\tilde{a}(\omega\cdot t) = \lim_{n\to\infty}\tilde{a}(\omega_n\cdot t) = \lim_{n\to\infty} a(\omega_n\cdot t) \leqslant \lim_{n\to\infty} u(t,\omega_n,a(\omega_n)) = \lim_{n\to\infty} u(t,\omega_n,\tilde{a}(\omega_n)) = u(t,\omega,\tilde{a}(\omega))$, so that $\tilde{a}$ is a subequilibrium. And it is also a superequilibrium, as deduced from the second inequality in (2.5), so that it is an equilibrium.

(iii) Let us write $\lambda_1 = l_{K^1}(K^\lambda)$. Since c^λ is a subequilibrium for $\lambda \in (0,1)$, $c^\lambda \leqslant K^\lambda$ and hence $\lambda \leqslant \lambda_1$. On the other hand, $K^\lambda = \{d^\lambda\}$ with $d^\lambda \geqslant c^{\lambda_1} \geqslant c^\lambda$. The monotonicity of the semiflow ensures that $K^\lambda = K^{\lambda_1}$ which together with the injectivity of the map $(0,1] \to \mathcal{P}_c(\Omega\times X)$, $\mu \mapsto K^\mu$ established in Proposition 3.7 in [26] guarantees that $\lambda = \lambda_1$.

(iv) The last properties follow easily from the definition of the label and (ii). Note that $c^\lambda \leqslant K^\lambda \leqslant K = \tilde{c}^\lambda$ and $c^\lambda(\omega) = \tilde{c}^\lambda(\omega)$ for $\omega \in R$ ensure that $K = K^\lambda$. $\square$

We will refer to c_i as the ith component of $K = \{c_1, c_2\}$. We say that $K = \{c_1, c_2\} \in \mathcal{M}_{a,K^1}$ is *labeled by λ on its ith component* if c_i and c_i^λ agree in a residual set of points. To unify notation, in the case that a_i is C^1 we represent $\tilde{a}_i = a_i$ and $\tilde{c}_i^\lambda = c_i^\lambda$. Note that: the functions $\tilde{a}_i$ and $\tilde{c}_i^\lambda$ exist (at least) when there exists $K \in \mathcal{M}_{a,K^1}$ labeled on its ith component; that they are the unique continuous functions which respectively agree with a_i and c_i^λ in a residual subset of Ω; that they are above a_i and c_i^λ, as deduced from the semicontinuity of a and c^λ; that they satisfy the conditions of Remark 2.7, as deduced from the previous proof; and that they agree with a_i and c_i^λ at least in the case that K is labeled only on this component. The set K is labeled on both components if and only if $K = \{\tilde{c}_1^\lambda, \tilde{c}_2^\lambda\}$ for a $\lambda \in (0,1)$, in which case $K = K^\lambda$.

Remark 3.3. For further purposes we point out that the previous concepts and results are valid in the case that the base of the skew-product semiflow is a minimal semiflow instead of a flow. The only point to have in mind is that any $\omega \in \Omega$ admits at least a backward orbit, which is dense, and which we choose and fix to prove point (ii) of Proposition 3.2.

The next auxiliary results contain the fundamental facts used later to describe the possible types of dynamics and the possible types of equations giving rise to them. Proposition 3.2 is repeatedly used in their proofs.

Proposition 3.4. *Assume that the family* (2.2), (2.3) *or* (2.4) *satisfies* (H1)–(H3), *and that there exists a minimal set $K \in \mathcal{M}_{a,K^1}$ labeled on its first component with $l_{K^1}(K) = \lambda_0 \in (0,1)$. Then,*

(i) *K^{λ_0} is labeled on its first component.*
(ii) *If $K^{\lambda_0} < K$ then a is a C^1 equilibrium, $K^{\lambda_0} = \{c^{\lambda_0}\}$, $K = \{c_1^{\lambda_0}, c_2^{\lambda_2}\}$ for $\lambda_2 > \lambda_0$, and $\{c_1^{\lambda_0}, c_2^\lambda\}$ is minimal for every $\lambda \in [\lambda_0, \lambda_2]$.*

(iii) *There exists $M \in \mathcal{M}_{a,K^1}$ with $l_{K^1}(M) = \lambda_0$ labeled on its second component if and only if $K^{\lambda_0} = \{\tilde c^{\lambda_0}\}$. In this case, $\tilde a$ is a C^1 equilibrium and for any $\omega \in \Omega$ there is $i = i(\omega) \in \{1,2\}$ with $\tilde a_i(\omega\cdot t) = a_i(\omega\cdot t)$ for any $t \in \mathbb{R}$.*

Symmetric conclusions hold if the initial set K is labeled on its second component.

Proof. (i) Let us write $K = \{\tilde c_1^{\lambda_0}, c_2\}$, with $c_2 = \tilde c_2^{\lambda_0}$ or $c_2 \gg c_2^{\lambda_0}$. Assertion (i) follows from $c^{\lambda_0} \leqslant K^{\lambda_0} \leqslant K$. In particular, $K^{\lambda_0} = \{\tilde c_1^{\lambda_0}, d_2\}$, with $c_2^{\lambda_0} \leqslant d_2 \leqslant c_2$.

(ii) In order to prove (ii), note that $K^{\lambda_0} < K$ ensures that $c_2 \gg c_2^{\lambda_0}$. In particular, a_1 is C^1 and $\tilde c_1^{\lambda} = c_1^{\lambda}$ for any $\lambda \in [0,1]$. According to Proposition 2.9, the marginal semiflow $\tau_{2,c_1^{\lambda_0}}$ satisfies the hypotheses of Theorem 3.13 in [28] both for the $\tau_{2,c_1^{\lambda_0}}$-subequilibria a_2 and $c_2^{\lambda_0}$ and for the minimal set $\{c_2\} \gg c_2^{\lambda_0} \geqslant a_2$. Since $\{d_2\} \gg a_2$ is a new minimal set, we deduce, on the one hand, the existence of a C^1 $\tau_{2,c_1^{\lambda_0}}$-equilibrium $\tilde a_2$ agreeing with a_2 at its continuity points and $\rho_1 \in (0,1)$ with $d_2 = \rho_1 c_2 + (1-\rho_1)\tilde a_2$; and, on the other hand, the existence of a C^1 $\tau_{2,c_1^{\lambda_0}}$-equilibrium $\tilde c_2^{\lambda_0}$ agreeing with $c_2^{\lambda_0}$ at its continuity points and $\rho_2 \in [0,1)$ with $d_2 = \rho_2 c_2 + (1-\rho_2)\tilde c_2^{\lambda_0}$. It is clear that $\tilde c_2^{\lambda_0} = \lambda_0 c_2^1 + (1-\lambda_0)\tilde a_2$, from where it follows that $c_2 = \tilde c_2^{\lambda_2}$ and $d_2 = \tilde c_2^{\lambda_1}$ for suitable $\lambda_2 > \lambda_1 \geqslant \lambda_0$, with $\tilde c_2^{\lambda} = \lambda c_2^1 + (1-\lambda)\tilde a_2$. In addition, and also according to Theorem 3.13 in [28], $\{\tilde c_2^{\lambda}\}$ is a $\tau_{2,c_1^{\lambda_0}}$-minimal set for any $\lambda \in [\lambda_0, \lambda_2]$. Now assume by contradiction that $a_2(\omega) < \tilde a_2(\omega)$ for a point $\omega \in \Omega$ and apply Proposition 2.4 to the points $(\omega, x) = (\omega, y) = (\omega, c_1^{\lambda_0}(\omega), \tilde a_2(\omega))$ and $\lambda = 0$ in order to conclude that $a_2(\omega_1) \ll \tilde a_2(\omega_1)$ for a point $\omega_1 \in \Omega$. Proposition 3.6(iii) in [28] applied to the $\tau_{2,c_1^{\lambda_0}}$-subequilibrium a_2 and the point $(\omega_1, \tilde a_2(\omega_1))$ then ensures that $a_2 \ll \tilde a_2$, impossible. So that $\tilde a_2 = a_2$ and $\tilde c_2^{\lambda} = c_2^{\lambda}$ for any $\lambda \in [0,1]$. Once known that $\tilde a = a$, it follows from Proposition 3.2(ii) that it is an equilibrium.

On the other hand, and reasoning for simplicity of the notation in the ODEs case, since F_1 increases with respect to x_2 and $(c_1^{\lambda_0})'(\omega) \leqslant F_1(\omega, c_1^{\lambda_0}(\omega), c_2^{\lambda_0}(\omega)) \leqslant F_1(\omega, c_1^{\lambda_0}(\omega), c_2^{\lambda_2}(\omega)) = (c_1^{\lambda_0})'(\omega)$, then $(c_1^{\lambda_0})'(\omega) = F_1(\omega, c_1^{\lambda_0}(\omega), c_2^{\lambda}(\omega))$ for every $\lambda \in [\lambda_0, \lambda_2]$. Altogether, these properties imply that $\{c_1^{\lambda_0}, c_2^{\lambda}\}$ is τ-minimal for every $\lambda \in [\lambda_0, \lambda_2]$. In turn, this fact and Proposition 3.2(iv) ensure that $K^{\lambda_0} = \{c^{\lambda_0}\}$, completing the proof. The same argument works in the delay and parabolic cases.

(iii) The first assertion in (iii) is an immediate consequence of (i): K^{λ_0} is simultaneously labeled on both components. Proposition 3.2(ii) shows that $\tilde a$ exists and is a C^1 equilibrium. Now we assume by contradiction the existence of $\omega_1 \in \Omega$ and $t_1 \in \mathbb{R}$ such that $a_1(\omega_1) < \tilde a_1(\omega_1)$ and $a_2(\omega_1\cdot t_1) < \tilde a_2(\omega_1\cdot t_1)$. We apply Proposition 2.4 twice: firstly to $(\omega_1, x_1) = (\omega_1, y_1) = (\omega_1, \tilde a(\omega_1))$, $i = 1$ and $\lambda = 1$; and secondly to $(\omega_1\cdot t_1, x_2) = (\omega_1\cdot t_1, y_2) = (\omega_1\cdot t_1, \tilde a(\omega_1\cdot t_1))$, $i = 2$ and $\lambda = 1$. The conclusion if that, for large enough t, $a(\omega_1\cdot t) \ll \tilde a(\omega_1\cdot t)$. Proposition 3.6(iii) in [28] then ensures that $a \ll \tilde a$, impossible. □

Proposition 3.5. *Assume that the family* (2.2), (2.3) *or* (2.4) *satisfies* (H1)–(H3), *and that there exists a minimal set $K \in \mathcal{M}_{a,K^1}$ different from K^1 and labeled on its first component. Then,*

(i) *for any $\lambda \in [0,1]$ and in the ODEs, delay and parabolic cases respectively,*

$$\big(\tilde c_1^{\lambda}\big)'(\omega) = F_1\big(\omega, \tilde c_1^{\lambda}(\omega), c_2^{\lambda}(\omega)\big),$$

$$\big(\bar{\tilde c}_1^{\lambda}\big)'(\omega) = F_1\big(\omega, \bar{\tilde c}_1^{\lambda}(\omega), \bar c_2^{\lambda}(\omega), \bar{\tilde c}_1^{\lambda}\big(\omega\cdot(-1)\big), \bar c_2^{\lambda}\big(\omega\cdot(-1)\big)\big),$$

$$\big(\bar{\tilde c}_1^{\lambda}\big)'(\omega, v) = d_1 \Delta \bar{\tilde c}_1^{\lambda}(\omega, v) + F_1\big(\omega, v, \bar{\tilde c}_1^{\lambda}(\omega, v), \bar c_2^{\lambda}(\omega, v)\big),$$

and c_2^{λ} can be replaced by $\tilde c_2^{\lambda}$ if $(\tilde a_1, \tilde a_2)$ exists and is a C^1 equilibrium.

(ii) *If K is labeled by λ_0 only on its first component, or if there exists $M \in \mathcal{M}$ with $M = \{\tilde{c}_1^{\lambda_0}, d_2\}$ for $d_2 \ll c_2^{\lambda_0}$, then*

$$\big(\tilde{c}_1^{\lambda_0}\big)'(\omega) = F_1\big(\omega, \tilde{c}_1^{\lambda_0}(\omega), x_2\big) \quad \textit{whenever } x_2 \geqslant c_2^{\lambda_0}(\omega),$$

$$\big(\bar{\tilde{c}}_1^{\lambda_0}\big)'(\omega) = F_1\big(\omega, \bar{\tilde{c}}_1^{\lambda_0}(\omega), x_2, \bar{\tilde{c}}_1^{\lambda_0}\big(\omega\cdot(-1)\big), x_4\big) \quad \textit{whenever } x_2 \geqslant \bar{c}_2^{\lambda_0}(\omega) \textit{ and } x_4 \geqslant \bar{c}_2^{\lambda_0}\big(\omega\cdot(-1)\big),$$

$$\big(\bar{\tilde{c}}_1^{\lambda_0}\big)'(\omega, v) = d_1 \Delta \bar{\tilde{c}}_1^{\lambda_0}(\omega, v) + F_1\big(\omega, v, \bar{\tilde{c}}_1^{\lambda_0}(\omega, v), x_2\big) \quad \textit{whenever } x_2 \geqslant \bar{c}_2^{\lambda_0}(\omega, v)$$

in the ODEs, delay and parabolic cases respectively.

(iii) *If there exists $M \in \mathcal{M}_{a,K^1}$ different from K^1 (maybe $M = K$) labeled on its second component, then $K^\lambda = \{\tilde{c}^\lambda\}$ for every $\lambda \in (0,1]$ and $\tilde{a}$ is a C^1 equilibrium. In addition, $\tilde{a}_i = a_i$ in the whole Ω for $i = 1$ or $i = 2$.*

In particular, K^λ is labeled on its first component for every $\lambda \in (0,1]$. Symmetric conclusions hold if the initial set K is labeled on its second component.

Proof. (i) Let us reason in the ODEs case: the other ones are analogous. Note that either a_1 is C^1 or $\tilde{a}$ exists and is a C^1 equilibrium, as proved in Proposition 3.2(ii). We assume first that a_1 is C^1 (so that $a_1 = \tilde{a}_1$), and write $K = \{c_1^{\lambda_0}, c_2\} = \{\tilde{c}_1^{\lambda_0}, c_2\}$ with $\lambda_0 \in (0,1)$ and $c_2 \geqslant c_2^{\lambda_0}$. By the increasing and concavity properties of F_1,

$$\begin{aligned}
\big(\tilde{c}_1^{\lambda_0}\big)'(\omega) &= F_1\big(\omega, \tilde{c}_1^{\lambda_0}(\omega), c_2(\omega)\big) \geqslant F_1\big(\omega, \tilde{c}_1^{\lambda_0}(\omega), c_2^{\lambda_0}(\omega)\big) \\
&\geqslant \lambda_0 F_1\big(\omega, c_1^1(\omega), c_2^1(\omega)\big) + (1-\lambda_0)\, F_1\big(\omega, \tilde{a}_1(\omega), a_2(\omega)\big) \\
&\geqslant \lambda_0 F_1\big(\omega, c_1^1(\omega), c_2^1(\omega)\big) + (1-\lambda_0)\tilde{a}_1'(\omega) = \big(\tilde{c}_1^{\lambda_0}\big)'(\omega).
\end{aligned}$$

The last inequality is ensured by Remark 2.7. We define $h_1 : \Omega \times [0,1] \to \mathbb{R}$ by $h_1(\omega,\lambda) = F_1(\omega, \tilde{c}_1^\lambda(\omega), c_2^\lambda(\omega)) - \lambda F_1(\omega, c_1^1(\omega), c_2^1(\omega)) - (1-\lambda)F_1(\omega, \tilde{a}_1(\omega), a_2(\omega))$, which due to the concavity of F_1 is nonnegative on its domain, and concave with respect to λ. Since $h_1(\cdot,0) = h_1(\cdot,\lambda_0) = h_1(\cdot,1) = 0$, necessarily $h_1(\cdot,\lambda) \equiv 0$ for every $\lambda \in [0,1]$, which implies (i).

In the case that $\tilde{a}$ exists and is a C^1 equilibrium, we can repeat the previous argument with a replaced by $\tilde{a}$ and with c_2^λ replaced by $\tilde{c}_2^\lambda$, since $c_2 \geqslant \tilde{c}_2^{\lambda_0}$. In order to check that in this last case the equalities hold both for c_2^λ and $\tilde{c}_2^\lambda$, we fix $\omega \in \Omega$. According to Proposition 3.4(iii), $a_i(\omega\cdot t) = \tilde{a}_i(\omega\cdot t)$ for every $t \in \mathbb{R}$ for an $i \in \{1,2\}$. In the case $i = 2$, $c_2^\lambda(\omega) = \tilde{c}_2^\lambda(\omega)$, so that the property holds. Assume now $i = 1$. Then, by the increasing character of F_1 with respect to its second state variable, $(\tilde{c}_1^\lambda)'(\omega) = F_1(\omega, \tilde{c}_1^\lambda(\omega), \tilde{c}_2^\lambda(\omega)) \geqslant F_1(\omega, \tilde{c}_1^\lambda(\omega), c_2^\lambda(\omega))$. On the other hand, Remark 2.7 ensures that $(\tilde{c}_1^\lambda)'(\omega) = (c_1^\lambda)'(\omega) \leqslant F_1(\omega, c_1^\lambda(\omega), c_2^\lambda(\omega)) = F_1(\omega, \tilde{c}_1^\lambda(\omega), c_2^\lambda(\omega))$. Both inequalities show our assertion.

(ii) Assume that $K = \{\tilde{c}_1^{\lambda_0}, d_2\}$ for $d_2 \gg c_2^{\lambda_0}$, in which case $\tilde{c}_1^{\lambda_0} = c_1^{\lambda_0}$. Again in the ODEs case, $(\tilde{c}_1^{\lambda_0})'(\omega) = F_1(\omega, \tilde{c}_1^{\lambda_0}(\omega), d_2(\omega)) = F_1(\omega, \tilde{c}_1^{\lambda_0}(\omega), c_2^{\lambda_0}(\omega))$, the second equality having been stated in (i). The concavity of F_1 and its increasing character with respect to x_2 prove the assertion. The same argument works if $M = \{\tilde{c}_1^{\lambda_0}, d_2\} \in \mathcal{M}$, with $d_2 \ll c_2^{\lambda_0}$. And the proofs in the delay and parabolic cases are identical.

(iii) In this situation, both $\tilde{c}_1^\lambda$ and $\tilde{c}_2^\lambda$ exist for every $\lambda \in [0,1]$, and their derivatives satisfy the corresponding equalities in (i). This implies that $K^\lambda = \{\tilde{c}^\lambda\}$ for $\lambda \in (0,1]$ and that $\tilde{a}$ is a C^1 equilibrium. Assume now the existence of $\omega_1 \in \Omega$ with $a_2(\omega_1) < \tilde{a}_2(\omega_1)$. Proposition 2.4 ensures that, if $t \geqslant 1$, $a_2(\omega_1\cdot t) \ll \tilde{a}_2(\omega_1\cdot t)$, or equivalently $c_2^\lambda(\omega_1\cdot t) \ll \tilde{c}_2^\lambda(\omega_1\cdot t)$ for every $\lambda \in [0,1)$. By (i), and again for ODEs, $(\tilde{c}_1^\lambda)'(\omega_1\cdot t) = F_1(\omega_1\cdot t, \tilde{c}_1^\lambda(\omega_1\cdot t), c_2^\lambda(\omega_1\cdot t)) = F_1(\omega_1\cdot t, \tilde{c}_1^\lambda(\omega_1\cdot t), \tilde{c}_2^\lambda(\omega_1\cdot t))$, which as in (ii) implies that $(\tilde{c}_1^\lambda)'(\omega_1\cdot t) = F_1(\omega_1\cdot t, \tilde{c}_1^\lambda(\omega_1\cdot t), x_2)$ whenever $x_2 \geqslant \tilde{c}_2^\lambda(\omega_1\cdot t)$. By continuity of all the functions involved, $(\tilde{c}_1^\lambda)'(\omega) = F_1(\omega, \tilde{c}_1^\lambda(\omega), x_2)$ whenever $x_2 \geqslant \tilde{c}_2^\lambda(\omega)$, which implies that $\{\tilde{c}_1^\lambda\}$ is $\tau_{1,\tilde{c}_2^\mu}$-minimal whenever $\lambda \in [0, \mu]$. This and Theorem 3.13 in [28] ensure that $a_1 \ll \{\tilde{a}_1\}$ if $a_1(\omega_2) \ll \tilde{a}_1(\omega_2)$ for a point $\omega_2 \in \Omega$, which according to Proposition 2.4 is the case if $a_1 < \tilde{a}_1$. But the strong inequality contradicts the coincidence of a_1 and $\tilde{a}_1$ in a residual subset of Ω, so that $a_1 = \tilde{a}_1$.

In order to check that under the hypotheses of this theorem K^λ is labeled on its first component for every $\lambda \in (0, 1)$, we first point out that the last assertion of the theorem is obvious. Now we assume by contradiction that there is $\lambda_0 \in (0, 1)$ with $K^{\lambda_0} = \{d_1, \tilde{c}_2^{\lambda_0}\}$ for $d_1 \gg \tilde{c}_1^{\lambda_0}$. By (i), and again for ODEs, $(\tilde{c}_1^{\lambda_0})'(\omega) = F_1(\omega, \tilde{c}_1^{\lambda_0}(\omega), \tilde{c}_2^{\lambda_0}(\omega))$, while (ii) (which holds for the second component in these conditions) ensures that $(\tilde{c}_2^{\lambda_0})'(\omega) = F_2(\omega, \tilde{c}_1^{\lambda_0}(\omega), \tilde{c}_2^{\lambda_0}(\omega))$. In other words, $\{\tilde{c}_1^{\lambda_0}, \tilde{c}_2^{\lambda_0}\}$ is minimal, which according to Proposition 3.2(iv) means that it agrees with K^{λ_0}. The proof is complete. □

We say that a label $\lambda \in (0, 1]$ is *single* (w.r.t. K^1) if $\mathcal{M}_{a,K^1}$ contains exactly one element labeled by λ; otherwise we say that it is *multiple*. Note that 1 is single. The next two subsections analyze the two different possibilities arising for the dynamics. The two situations described by Theorems 3.7 and 3.12 exhaust the possibilities for the global dynamics if the area strongly above any given subequilibrium contains more than two minimal sets.

As in the previous results, when the proofs of the next subsections require working with the differential equations giving rise to the semiflow, we will always work in the ODEs case. This is uniquely due to the simplicity of the notation: the subjacent ideas are identical, and any small change, if needed, will be indicated in due time.

3.1. Dynamics in the multiple-labeling case

The first result in this paragraph gives a complete description of the set $\mathcal{M}_{a,K^1}$ when a multiple label exists, situation in which a is necessarily a C^1 equilibrium. Once this is done, in the second theorem, we derive the shape of the set $\mathcal{M}$ of all the minimal sets.

As in Definition 2.2, a continuous function $h : \mathbb{R}^2 \to \mathbb{R}$ is *concave* if for any $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}^2$ with $x \leqslant y$ it is $h(\lambda y + (1 - \lambda)x) \geqslant \lambda h(y) + (1 - \lambda)h(x)$. It is easy to check that, in this case, the restriction of h to any line joining two ordered points $x \leqslant y$ in $\mathbb{R}^2$, i.e., the function $h_{x,y} : \mathbb{R} \to \mathbb{R}$, $\lambda \mapsto h(\lambda y + (1 - \lambda)x)$, is also concave. Consequently, if $h_{x,y}$ vanishes at two different points $\lambda_1 < \lambda_2$, then it is nonnegative in (λ_1, λ_2) and nonpositive in $\mathbb{R} - [\lambda_1, \lambda_2]$. This property will be repeatedly used in what follows.

Theorem 3.6. *Assume that the family* (2.2)*,* (2.3) *or* (2.4) *satisfies* (H1)–(H3)*, and that the dynamics above a fits Case* A2 *of Theorem* 3.8 *in* [28]*. Assume also the existence of a reference minimal set* $K^1 = \{c_1^1, c_2^1\}$ *for which there exists a multiple label. Then, a is a* C^1 *equilibrium, any label of* $(0, 1)$ *is multiple, and any element of* $\mathcal{M}_{a,K^1}$ *is of the form* $\{\alpha_1 c_1^1 + (1 - \alpha_1)a_1, \alpha_2 c_2^1 + (1 - \alpha_2)a_2\}$ *where* (α_1, α_2) *varies on a convex set* $\mathcal{F}^1 \subset (0, 1] \times (0, 1]$ *containing* $\{(\lambda, \lambda) \mid \lambda \in (0, 1]\}$ *and which is contained in the closure of its interior.*

Proof. According to Proposition 3.2 (iii) and (iv), the existence of a multiple label $\lambda_0 \in (0, 1)$ ensures the existence of $K > K^{\lambda_0}$ labeled by λ_0. Consequently, Proposition 3.4(ii) ensures that a is a C^1 equilibrium and allows us to assume without restriction the existence of $\lambda_2 > \lambda_0$ such that $\{c_1^{\lambda_0}, c_2^\lambda\}$ is minimal for every $\lambda \in [\lambda_0, \lambda_2]$. We reason in the ODEs case, in which $a_i'(\omega) = F_i(\omega, a_1(\omega), a_2(\omega))$. Let us define the functions $g_i : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, 2$ by $g_i(\omega, \alpha_1, \alpha_2) = F_i(\omega, c_1^{\alpha_1}(\omega), c_2^{\alpha_2}(\omega)) - \alpha_i F_i(\omega, c_1^1(\omega), c_2^1(\omega)) - (1 - \alpha_i)F_i(\omega, a_1(\omega), a_2(\omega))$. In other words, $g_i(\omega, \alpha_1, \alpha_2) = F_i(\omega, c_1^{\alpha_1}(\omega), c_2^{\alpha_2}(\omega)) - (c_i^{\alpha_i})'(\omega)$. Due to the concavity of F_i, g_i is concave in (α_1, α_2) (in the order-concave sense of Definition 2.2). Note that $(c_1^{\alpha_1}, c_2^{\alpha_2})$ determines an equilibrium if and only if $g_i(\cdot, \alpha_1, \alpha_2) \equiv 0$ for $i = 1, 2$. In particular, $g_i(\cdot, \lambda, \lambda) \equiv 0$ for every $\lambda \in [0, 1]$, since Proposition 3.5(iii) ensures that $K^\lambda = \{c_1^\lambda, c_2^\lambda\}$ for any $\lambda \in (0, 1]$. And, as seen above, $g_i(\cdot, \lambda_0, \lambda) \equiv 0$ for any $\lambda \in [\lambda_0, \lambda_2]$.

Let us check that g_1 and g_2 vanish identically at every point of the triangle T with vertices $(0, 0)$, $(1, 1)$ and (λ_0, λ_2). We begin with g_2, which increases with respect to α_1. Consequently, since it vanishes at (λ, λ) for $\lambda \in [0, 1]$, it is nonpositive at the whole T; and, since in addition $g_2(\cdot, \lambda_0, \lambda) =$

$g_2(\cdot,\lambda,\lambda)\equiv 0$ for any $\lambda\in[\lambda_0,\lambda_2]$, then $g_2(\cdot,\mu,\lambda)\equiv 0$ for any (μ,λ) in the triangle with vertices (λ_0,λ_0), (λ_0,λ_2) and (λ_2,λ_2). Now we deduce from its concavity that g_2 is nonpositive in T: any point in the triangle is the convex combination of two ordered points in T for which g_2 vanishes, the first one in the diagonal and the second one in the segment joining (λ_0,λ_0) with (λ_0,λ_2). So that it vanishes at any point of T. Let us now analyze the function g_1, which due to its increasing character with respect to α_2 is nonnegative in T. Since $g_1(\cdot,\lambda_0,\lambda)=g_1(\cdot,\lambda,\lambda)\equiv 0$ for any $\lambda\in[\lambda_0,\lambda_2]$, it is nonpositive and hence null at the points (μ,λ) with $\lambda\in[\lambda_0,\lambda_2]$ and $0\leqslant\mu\leqslant\lambda_0$, which also ensures that it is nonnegative and hence null in the rest of points of T.

For further purposes we point out that g_1 and g_2 are nonpositive in the whole plane. Let us check this assertion for g_1. Since it vanishes in T and is concave and increasing in α_2, it vanishes in $\{(\alpha_1,\alpha_2)\mid 0\leqslant\alpha_1\leqslant 1 \text{ and } \alpha_2\geqslant\alpha_1\}$. Again by concavity, it is nonpositive whenever $\alpha_1\in\mathbb{R}$ and $\alpha_2\geqslant 1$, and hence its increasing character on α_2 ensures the stated property.

In particular, each label $\lambda\in(0,1)$ is multiple. This and Proposition 3.4 (i) and (ii) lead us to ensure that the elements of $\mathcal{M}_{a,K^1}$ are in one-to-one correspondence with the points of the set $\mathcal{F}^1=\{\alpha=(\alpha_1,\alpha_2)\in(0,1]\times(0,1]\mid g_i(\cdot,\alpha_1,\alpha_2)\equiv 0 \text{ for } i=1,2\}$, which contains T. In addition, $\mathcal{F}^1$ is convex and contained in the closure of its interior, as easily deduced from the fact that any one of its points outside the diagonal of $\mathbb{R}^2$ determines, together with $(0,0)$ and $(1,1)$ a triangle contained in it. In turn, this property follows again from the concavity, the nonpositive character and the increasing properties of g_i.

This completes the proof for ODEs. In the delay and parabolic cases, we define

$$g_i(\omega,\alpha_1,\alpha_2)=F_i\big(\omega,\bar c_1^{\alpha_1}(\omega),\bar c_2^{\alpha_2}(\omega),\bar c_1^{\alpha_1}\big(\omega\cdot(-1)\big),\bar c_2^{\alpha_2}\big(\omega\cdot(-1)\big)\big)-\big(\bar c_i^{\alpha_i}\big)'(\omega),$$
$$g_i(\omega,v,\alpha_1,\alpha_2)=d_i\Delta\bar c_i^{\alpha_i}(\omega,v)+F_i\big(\omega,v,\bar c_1^{\alpha_1}(\omega,v),\bar c_2^{\alpha_2}(\omega,v)\big)-\big(\bar c_i^{\alpha_i}\big)'(\omega,v)$$

respectively, and apply the same arguments as above. □

We say that a set $\mathcal{F}\subset\mathbb{R}^2$ is *order-convex* if $\lambda x+(1-\lambda)y\in\mathcal{F}$ for $\lambda\in[0,1]$ whenever $x,y\in\mathcal{F}$ and $x\leqslant y$.

Theorem 3.7. *Assume that the family* (2.2), (2.3) *or* (2.4) *satisfies* (H1)–(H3), *and that the dynamics above a fits Case* A2 *of Theorem* 3.8 *in* [28]. *Assume also that there exists a reference minimal set* $K^1=\{c_1^1,c_2^1\}\gg a$ *for which there exists a multiple label. Then,*

- (i) *Any minimal set is of the form* $\{\alpha_1 c_1^1+(1-\alpha_1)a_1,\alpha_2 c_2^1+(1-\alpha_2)a_2\}$ *where* $\alpha=(\alpha_1,\alpha_2)$ *varies in a closed order-convex set* $\mathcal{F}\subset\mathbb{R}^2$ *with nonempty interior containing* $\{(\lambda,\lambda)\mid\lambda\in[0,1]\}$.
- (ii) *The set* $\mathcal{F}$ *is bounded from above if and only if there exists a top minimal set strongly above a, or equivalently, if and only if there exists* $\alpha^+\in\mathcal{F}$ *such that* $\alpha\leqslant\alpha^+$ *for any* $\alpha\in\mathcal{F}$. *In this case,* $\{c_1^{\alpha_1^+},c_2^{\alpha_2^+}\}$ *is the top minimal set of the semiflow.*
- (iii) *If* $\mathcal{F}$ *is unbounded from above, for any* $\alpha\in\mathcal{F}$ *there exists* $\bar\alpha\in\mathcal{F}$ *with* $\bar\alpha>\alpha$.
- (iv) *For any* $\alpha_*\in\mathcal{F}$, *the sets* $\mathcal{F}^{\alpha_*}=\{\alpha\in\mathcal{F}\mid\alpha\gg\alpha_*\}$ *and* $\mathcal{F}_{\alpha_*}=\{\alpha\in\mathcal{F}\mid\alpha\ll\alpha_*\}$, *if nonempty, are contained in the closures of their interiors.*

Proof. The key point for the proof is to show the first assertion in (i); that is,

(m) any minimal set is of the form $K=\{c_1^{\alpha_1},c_2^{\alpha_2}\}$,

where the definition of c^λ given by (3.1) extends to $\lambda\in\mathbb{R}$. Let M be a τ-minimal set. The monotonicity of the semiflow ensures that $M\leqslant K^{\lambda_1}$ for $\lambda_1\geqslant 1$ large enough, with K^{λ_1} defined by (3.1). We fix such a λ_1, take K^{λ_1} as new reference minimal set, and rename it by $K^1=\{c^1\}$. Note that this causes no essential change in assertion (m), since the former K^1 is one of the minimal sets described in Theorem 3.6 for the new reference. Assume first that $M\ll K^1$. Theorem 3.6 ensures the existence of

$\lambda \in (0,1)$ with $M \ll c^\lambda \ll K^1$. If, in addition, M is a copy of the base, say $M = \{d\}$, we conclude again from Theorem 3.6 (applied to the new K^1 and to d instead of a) that $c_i^\lambda(\omega) = \mu_i c_i^1(\omega) + (1-\mu_i)d_i(\omega)$, so that $d_i = c_i^{\alpha_i}$ for suitable α_i, $i = 1,2$, and hence $M = \{c_1^{\alpha_1}, c_2^{\alpha_2}\}$. But, of course, we cannot assume a priori that M is a copy of the base. To cope with this technical difficulty we consider an extension of the semiflow τ to the product space $M \times X$, by defining, for instance in the delay case, $G_i : M \times \mathbb{R}^4 \to \mathbb{R}$ by $G_i(\omega, z, x_1, x_2, x_3, x_4) = F_i(\omega, x_1, x_2, x_3, x_4)$ for $(\omega, z) \in M$ and $i = 1,2$, which satisfy the corresponding properties (H1)–(H3). The semiflow τ^M induced on $M \times X$ by the corresponding family of delay equations is given by $\tau^M(t,\omega,z,x) = (\omega\cdot t, u(t,\omega,z), u(t,\omega,x))$. It is clear that $K_*^1 = \{(\omega, z, c^1(\omega)) \mid (\omega, z) \in M\}$ and $M_* = \{(\omega, z, z) \mid (\omega, z) \in M\}$ are C^1 τ^M-equilibria: $M_* = \{d^*\}$, with $d^*(\omega, z) = z$. Now we can apply the previous argument in order to conclude that $d_i^*(\omega, z) = c_i^{\alpha_i}(\omega)$ for $i = 1,2$, and hence that $M = \{c_1^{\alpha_1}, c_2^{\alpha_2}\}$. Thus, (m) is proved in this situation.

Now assume $M < K^1$, but $M \not\ll K^1$. Repeating the arguments of Proposition 3.2(ii), we show that all the points of M are simultaneously of the form $(\omega, c_1^1(\omega), x_2)$ with $x_2 \ll c_2^1(\omega)$ or of the form $(\omega, x_1, c_2^1(\omega))$ with $x_1 \ll c_1^1(\omega)$. Assume without restriction that c_1^1 is the first fiber component of M. This means that both $\{c_2^1\}$ and $M_2 = \{(\omega, x_2) \mid (\omega, c_1^1(\omega), x_2) \in M\}$ are different minimal sets for the marginal semiflow τ_{2,c_1^1}. In addition, if follows from Theorem 3.6 and Proposition 3.5(ii) that $\{c_2^\lambda\}$ is a τ_{2,c_1^1}-minimal set for any $\lambda \in (0,1)$. Theorem 3.13 in [28] then ensures that $M_2 = \{c_2^\lambda\}$ for a $\lambda \leqslant 1$, and this completes the proof of (m).

This means that the set of minimal sets is in one-to-one correspondence with the elements of $\mathcal{F} = \{\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \mid g_i(\cdot, \alpha_1, \alpha_2) \equiv 0 \text{ for } i = 1,2\}$, with g_i defined in the previous proof, which as seen there has nonempty interior. Here we recover the initial set K^1. (The definition of $\mathcal{F}$ must be adapted to the domain of g_i in the parabolic case.) Since, as also proved there, g_1 and g_2 are concave and nonpositive, $\mathcal{F}$ contains the segment joining any ordered pair of its points. This completes the proof of (i).

Properties (ii) and (iii) are easy consequences of Remark 3.9.1 in [28] and from the existence for any $M \in \mathcal{M}$ of $\lambda \geqslant 1$ with $M \leqslant K^\lambda$. And (iv) follows from Theorem 3.6, with the two minimal sets corresponding to $\alpha_* \in \mathcal{F}$ and $\alpha \gg \alpha_*$ or $\alpha \ll \alpha_*$ playing the roles of initial (sub)equilibrium and minimal set strongly above it. The proof is complete. □

Let us define $\sigma : \mathbb{R} \to \mathbb{R}$ by $\sigma(x) = -x^2$ for $x \leqslant 0$ and $\sigma(x) = 0$ for $x \geqslant 0$. One of the simplest non-trivial examples of the multiple-labeling situation is given by the two-dimensional autonomous ODE $x_1' = \sigma(x_1)$, $x_2' = \sigma(x_2)$: the equilibrium points fill the positive cone of $\mathbb{R}^2$. But in fact it is very simple to find more sophisticated autonomous examples. For instance, take two C^1 maps $f, g : \mathbb{R} \to \mathbb{R}$, with f convex and g concave, strictly increasing and bijective, $f(0) = g(0) = 0$ and $f(1) < 1 \leqslant g(1)$. Then consider the system of ODEs $x_1' = \sigma(x_2 - f(x_1))$, $x_2' = \sigma(x_1 - h(x_2))$, where h is the inverse map of g, convex. It is obvious that it satisfies (H1) and it is easy to check that also (H2) holds. In addition, $(0,0)$ and $(1,1)$ are equilibrium points, which guarantees (H3). The set of equilibrium points is $\mathcal{F} = \{(x_1, x_2) \in \mathbb{R} \mid f(x_1) \leqslant x_2 \leqslant g(x_1)\} \subset \mathbb{R}_+ \times \mathbb{R}$. Note that if $f(x_*) = g(x_*)$ for a (unique) point $x_* > 1$, then $(x_*, f(x_*))$ is the top minimal set, while if $f < g$ in $(0, \infty)$, the top minimal set does not exist. Taking for instance $f(x) = x^2 - 2x$ and $g(x) = x$, we obtain an example in which $\mathcal{F}$ is bounded also from below, but without lowest minimal set.

An example of two-dimensional autonomous ODE for which the set $\mathcal{F}$ of equilibria is not a convex subset of $\mathbb{R}^2$ is provided by $x_1' = F_1(x_1, x_2)$, $x_2' = 0$, where

$$F_1(x_1, x_2) = \begin{cases} -2x_1 + x_1^2 x_2 + 1/x_2 & \text{if } x_2 \in (-\infty, 0) \text{ and } x_1 \leqslant 1/x_2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

It is a nice exercise to check that hypotheses (H1)–(H3) are fulfilled. In order to show that F_1 is concave (in the order-concave sense of Definition 2.2) it suffices to check that the second order derivative of the map $\lambda \to F_1(x_1 + \lambda v_1, x_2 + \lambda v_2)$ is negative if $(v_1, v_2) > 0$ while $(x_1 + \lambda v_1, x_2 + \lambda v_2) \in D = \{(x_1, x_2) \mid x_2 \in (-\infty, 0) \text{ and } x_1 \leqslant 1/x_2\}$. The corresponding set $\mathcal{F}$ is $\mathbb{R}^2 - D$, and as said above it is not convex: it does not contain any point in the segment joining two points of the border of D.

3.2. Dynamics in the single-labeling case

In this paragraph we describe the set $\mathcal{M}_a$, and then give some fundamental keys in the more complex description of $\mathcal{M}$. Autonomous examples show that the description is exhaustive, although the number of possible different situations is quite high.

Before stating the first theorem, we explain some properties which will be fundamental in its proof. Once $K^1 \gg a$ is fixed, we define c^λ for any $\lambda \in \mathbb{R}$ and K^λ for any $\lambda > 0$ by (3.1), and denote by $J \supseteq (0,1]$ the injectivity interval of the map $(0,\infty) \to \mathcal{P}_c(\Omega \times X)$, $\lambda \mapsto K^\lambda$ provided by Proposition 3.7 in Case A2 of Theorem 3.8 in [28]. According to Proposition 3.6 in [28], since c^λ is strongly above a for every $\lambda > 0$, $u(t,\omega,c^\lambda(\omega))$ exists for every $t \geqslant 0$ and the omega-limit set K^λ is strongly above a. In addition, it follows from (2.5) that c^λ is a superequilibrium for any $\lambda > 1$ and hence, by monotonicity, $a \ll K^\lambda \leqslant c^\lambda$. In addition, c^λ is continuous at the same points as a. We assume that $\sup J > 1$ and, for $\lambda > 1$ in J, we define $\mu(\lambda)$ as the minimum value of μ such that $K^\lambda \leqslant c^\mu$. Then $\mu(\lambda) = \lambda$: since $K^\lambda \leqslant c^\lambda$, $\mu(\lambda) \leqslant \lambda$; and, due to the monotonicity of the semiflow, the inequality $\mu(\lambda) < \lambda$ would contradict the injectivity of $\lambda \mapsto K^\lambda$ in $[\mu(\lambda),\lambda] \subset J$. As in Proposition 3.2(ii), we deduce that K^λ is either $\{\tilde{c}^\lambda\}$ (in which case $\tilde{a}$ is a C^1 equilibrium), or $\{c_1^\lambda, d_2^\lambda\}$ with $c_1^\lambda = \tilde{c}_1^\lambda$ and $d_2^\lambda \ll c_2^\lambda$ (in which case a_1 is C^1), or $\{d_1^\lambda, c_2^\lambda\}$ with $c_2^\lambda = \tilde{c}_2^\lambda$ and $d_1^\lambda \ll c_1^\lambda$ (in which case a_2 is C^1). We say that K^λ is labeled w.r.t. K^1 on its first (or second) component if this one is $\tilde{c}_1^\lambda$ (or $\tilde{c}_2^\lambda$), and write $l_{K^1}(K^\lambda) = \lambda$. Note that if $\lambda > 1$ and we take K^λ as new reference minimal set, then $l_{K^\lambda}(K^1) = 1/\lambda$, as easily deduced from the three possibilities for K^λ.

Proposition 3.8. *Assume that the family* (2.2), (2.3) *or* (2.4) *satisfies* (H1)–(H3), *and that the dynamics above a fits Case* A2 *of Theorem* 3.8 *in* [28]. *Assume also the existence of a reference minimal set $K^1 = \{c^1\} = \{c_1^1, c_2^1\} \gg a$ for which there exists a single label. Then,*

(i) *if $\lambda_* \in J - \{1\}$ and K^{λ_*} is labeled on its first* (*resp. second*) *component w.r.t. K^1, then K^λ is also labeled on its first* (*resp. second*) *component w.r.t. K^{λ_*} for any $\lambda \in (0,\lambda_*)$, and $l_{K^{\lambda_*}}(K^\lambda) = l_{K^1}(K^\lambda)/l_{K^1}(K^{\lambda_*}) = \lambda/\lambda_*$.*

(ii) *If in addition, K^{λ_*} is labeled only on its first* (*resp. second*) *component w.r.t. K^1, the same happens with K^λ for any $\lambda \in J - (0,\lambda_*)$, $\lambda \neq 1$.*

Proof. (i) The trivial equality $c^\lambda = (\lambda/\lambda_*)c^{\lambda_*} + (1-\lambda/\lambda_*)a$ for any pair λ, λ_* plays an important role in what follows. We also recall that $l_{K^1}(K^\lambda) = \lambda$ for any $\lambda \in J$.

We fix $\lambda_* \in J$ and represent $K^{\lambda_*} = \{d\} = \{d_1, d_2\}$. In the case that $d = \tilde{c}^{\lambda_*}$, Proposition 3.5(iii) (applied to the labeling w.r.t. K^1 if $\lambda_* < 1$ or w.r.t. K^{λ_*} otherwise) and the previous equality ensure that $K^\lambda = \{\tilde{c}^\lambda\} = \{(\lambda/\lambda_*)d + (1-\lambda/\lambda_*)\tilde{a}\}$ for any $\lambda \in (0,\lambda_*]$. The properties follow immediately from here.

In the rest of the proof of (i) we assume that K^{λ_*} is labeled only on its first component w.r.t. K^1, which means that $d_1 = c_1^{\lambda_*}$. We first consider the case $\lambda_* < 1$. According to Proposition 3.5(iii), this means that $K^\lambda = \{c_1^\lambda, d_2^\lambda\}$ with $d_2^\lambda \gg c_2^\lambda$ for any $\lambda \in (0,1)$. We fix $\lambda_* \in (0,1)$ and denote $\mu_\lambda = l_{K^{\lambda_*}}(K^\lambda)$. If K^λ is labeled w.r.t. K^{λ_*} on its first component, then $c_1^\lambda = \mu_\lambda c_1^{\lambda_*} + (1-\mu_\lambda)a_1$, and again the mentioned equality proves the result. We now work in the case in which K^λ is labeled w.r.t. K^{λ_*} on its second component; i.e., $K^\lambda = \{c_1^\lambda, \mu_\lambda d_2 + (1-\mu_\lambda)\tilde{a}_2\}$ with $c_1^\lambda \geqslant c_1^{\mu_\lambda\lambda_*}$. We will check that in this case $\{c_1^{\mu_\lambda\lambda_*}, \mu_\lambda d_2 + (1-\mu_\lambda)\tilde{a}_2\}$ is a minimal set. Since it is labeled on both components w.r.t. K^{λ_*}, the same happens for K^λ and hence $\lambda = \mu_\lambda\lambda_*$, as asserted. Proposition 3.5(ii) ensures that $(c_1^\mu)'(\omega) = F_1(\omega, c_1^\mu(\omega), x_2)$ if $x_2 \geqslant c_2^\mu(\omega)$ and $\mu \in (0,1)$, so that

$$\big(c_1^{\mu_\lambda\lambda_*}\big)'(\omega) = F_1\big(\omega, c_1^{\mu_\lambda\lambda_*}(\omega), \mu_\lambda d_2(\omega) + (1-\mu_\lambda)\tilde{a}_2(\omega)\big),$$

since $d_2 \gg c_2^{\lambda_*}$ and hence $\mu_\lambda d_2 + (1-\mu_\lambda)\tilde{a}_2 \gg c_2^{\mu_\lambda\lambda_*}$. Applying now Proposition 3.5(i) to the labeling w.r.t. K^{λ_*} we conclude that

$$\big(\mu_\lambda d_2 + (1-\mu_\lambda)\tilde{a}_2\big)'(\omega) = F_2\big(\omega, c_1^{\mu_\lambda\lambda_*}(\omega), \mu_\lambda d_2(\omega) + (1-\mu_\lambda)\tilde{a}_2(\omega)\big),$$

since $c_1^{\mu_\lambda\lambda_*} = \mu_\lambda c_1^{\lambda_*} + (1-\mu_\lambda)a_1$. These two equalities prove the assertion.

Finally, we consider the case $\lambda_* \in J - (0,1]$, which means that $d_2 \ll c_2^{\lambda_*}$. It is simple to check that K^1 is labeled only on its first component w.r.t. K^{λ_*}, with $l_{K^{\lambda_*}}(K^1) = 1/l_{K^1}(K^{\lambda_*})$. Note that in this case K^λ is labeled only on its first component w.r.t. K^{λ_*} for any $\lambda \in (0,\lambda_*)$. Assume first that $\lambda \in (0,1]$. By interchanging the roles of λ_* and 1 we deduce from the previous paragraph that $l_{K^1}(K^\lambda) = l_{K^{\lambda_*}}(K^\lambda)/l_{K^{\lambda_*}}(K^1) = l_{K^{\lambda_*}}(K^\lambda) l_{K^1}(K^{\lambda_*})$ for any $\lambda \in (0,1]$. Finally, if $\lambda \in (1,\lambda_*]$, we make $1, \lambda$ and λ_* respectively play the roles of λ, λ_* and 1, in order to conclude that $l_{K^\lambda}(K^1) = l_{K^{\lambda_*}}(K^1)/l_{K^{\lambda_*}}(K^\lambda)$ and hence that $l_{K^{\lambda_*}}(K^\lambda) = l_{K^{\lambda_*}}(K^1)/l_{K^\lambda}(K^1) = l_{K^1}(K^\lambda)/l_{K^1}(K^{\lambda_*})$, as asserted.

(ii) Take $\lambda \in J - (0,\lambda_*)$, $\lambda \neq 1$, and assume by contradiction that K^λ is labeled on its second component w.r.t. K^1. If $\lambda < 1$, Proposition 3.5 shows that K^{λ_*} is also labeled on its second component w.r.t. K^1. If $\lambda > 1 > \lambda_*$, since K^1 is labeled on its second component w.r.t. K^λ, (i) implies that K^{λ_*} is labeled in on its second component w.r.t. K^1. Finally, if $1 < \lambda_* < \lambda$, Proposition 3.5 ensures that K^{λ_*} is labeled on its second component w.r.t. K^λ, so that (i) implies that K^1 is labeled on its second component w.r.t. K^{λ_*}. In the three cases, we find the searched contradiction. □

Theorem 3.9. *Assume that the family* (2.2), (2.3) *or* (2.4) *satisfies* (H1)–(H3), *and that the dynamics above a fits Case* A2 *of Theorem* 3.8 *in* [28]. *Assume also the existence of a reference minimal set $K^1 = \{c^1\} = \{c_1^1, c_2^1\} \gg a$ for which there exists a single label. Then $\mathcal{M}_a = \{K^\lambda \mid \lambda \in J\}$ with $K^\lambda = \mathcal{O}(\omega, \lambda c^1(\omega) + (1-\lambda)a(\omega))$ for any $\omega \in \Omega$ and $K^{\lambda_1} < K^{\lambda_2}$ if $\lambda_1 < \lambda_2$, where $J \subseteq (0,\infty)$ is an interval satisfying $(0,1] \subset J$. In addition, $\sup J = \lambda_+ < \infty$ if and only if K^{λ_+} is the top minimal set in $\mathcal{M}_a$. Moreover, one of the three following possibilities holds*:

(a) *$K^\lambda = \{\tilde{c}_1^\lambda, \tilde{c}_2^\lambda\}$ for every $\lambda \in J$, with $\tilde{c}^\lambda = c^\lambda$ in a residual subset of Ω. In this case, there exists a C^1 equilibrium $\tilde{a}$ agreeing with a in a residual subset of Ω, with $\tilde{a}_1 = a_1$ or $\tilde{a}_2 = a_2$ in the whole Ω.*

(b) *$K^\lambda = \{c_1^\lambda, d_2^\lambda\}$ with $J \to X_2$, $\lambda \mapsto d_2^\lambda(\omega)$ continuous, increasing and concave for every $\omega \in \Omega$ and $d_2^{\lambda_*}(\omega) \neq c_2^{\lambda_*}(\omega)$ for a $\lambda_* \in J$ and every $\omega \in \Omega$. In this case, a_1 is C^1 in Ω with $a_1(\omega\cdot t) = u_1(t,\omega,a(\omega))$ for $\omega \in \Omega$ and $t \geqslant 0$. In addition, $J_1 = \{\lambda \in J \mid c_2^\lambda = d_2^\lambda$ in a residual subset of $\Omega\}$ is a proper right-closed subinterval of J which either reduces to $\{1\}$ or contains $(0,1]$, and $d_2^\lambda \gg c_2^\lambda$ and $d_2^\lambda \ll c_2^\lambda$ respectively in the first and second connected components of $J - J_1$ (one of them possibly empty). Moreover, $\{d_2^\lambda\}$ is the unique minimal set strongly above a_2 for the marginal semiflow τ_{2,c_1^λ} if $0 < \lambda < \sup J$, and the top one if $\lambda = \sup J < \infty$. Finally, either $d_2^{\lambda_1} = d_2^\lambda$ whenever $0 < \lambda_1 < \lambda \leqslant \sup J$ or $K^{\lambda_1} \ll K^\lambda$ whenever $0 < \lambda_1 < \lambda \leqslant \sup J$.*

(c) *$K^\lambda = \{d_1^\lambda, c_2^\lambda\}$ with $J \to X_1$, $\lambda \mapsto d_1^\lambda(\omega)$ continuous, increasing and concave for every $\omega \in \Omega$ and $d_1^{\lambda_*}(\omega) \neq c_1^{\lambda_*}(\omega)$ for a $\lambda_* \in J$ and every $\omega \in \Omega$. The symmetric properties to those of* (b) *hold.*

Proof. Let J be the injectivity interval of the map $\lambda \mapsto K^\lambda$ considered in the paragraph previous to Proposition 3.8. It is clear that every label with respect to any K^{λ_*} for $\lambda_* \in J$ is single: otherwise we would deduce from the shape of the set $\mathcal{F}$ of Theorem 3.7 the existence of multiple labels w.r.t. K^1, and hence that anyone of these labels is multiple, contradicting our hypothesis.

We already know that $K^{\lambda_1} < K^{\lambda_2}$ if $\lambda_1 < \lambda_2$ belong to J, and that $\sup J = \lambda_+ < \infty$ if and only if K^{λ_+} is the top minimal set in $\mathcal{M}_a$, as explained in Remark 3.9.1 in [28]. Let us check that $\mathcal{M}_a = \{K^\lambda \mid \lambda \in J\}$. Given $M \in \mathcal{M}_a$, we look for $\lambda_* \in J$ with $M \leqslant K^{\lambda_*}$, and represent $\mu = l_{K^{\lambda_*}}(M)$. Proposition 3.8 ensures that $l_{K^{\lambda_*}}(K^{\mu\lambda_*}) = \mu$ and hence $M = K^{\mu\lambda_*}$.

Let us define $J_1 = \{\lambda \in J \mid K^\lambda = \{\tilde{c}^\lambda\}\}$. Note that $1 \in J_1$. Assume first that the first condition in (a) holds: $J = J_1$. In this case, the stated properties of a have already been proved in Propositions 3.2(ii) and 3.5(iii).

For the rest of the proof we assume that $J_1 \neq J$ and that there exists K^{λ_0} labeled only on its first component. We will check that then (b) holds. It follows from Proposition 3.8 that $K^\lambda = \{c_1^\lambda, d_2^\lambda\}$ for any $\lambda \in J$. In addition, Proposition 3.5(iii) ensures that $J_1 \cap (0,1]$ either agrees with $(0,1]$ or reduces to $\{1\}$, and that $(0,\lambda_1] \subset J_1$ if $\lambda_1 \in J_1$ and $\lambda_1 \neq 1$. Consequently, either $J_1 = \{1\}$ or $J_1 = (0,\lambda_0]$ for $1 \leqslant \lambda_0 < \sup J$. We already know that $d_2^\lambda \gg c_2^\lambda$ and $d_2^\lambda \ll c_2^\lambda$ in the first (maybe empty) and second connected components of $J - J_1$ respectively, as well as that a_1 is C^1. In addition, $c_1^\lambda(\omega\cdot t) =$

$u_1(t,\omega,c_1^\lambda(\omega),d_2^\lambda(\omega)) \geqslant u_1(t,\omega,c_1^\lambda(\omega),c_2^\lambda(\omega))$ for $\omega\in\Omega$, $t\geqslant 0$ and $\lambda\in(0,1]$, and taking limit as $\lambda\to 0^+$, we obtain $a_1(\omega\cdot t)\geqslant u_1(t,\omega,a_1(\omega),a_2(\omega))\geqslant a_1(\omega\cdot t)$: the stated equality holds.

Let us check that d_2^λ is the unique minimal set strongly above a_2 for the marginal semiflow τ_{2,c_1^λ} for $0<\lambda<\sup J$. Proposition 3.8 ensures that K^λ is labeled w.r.t. K^{λ_*} by λ/λ_* only on its first component if $\lambda_*\in J$ is large enough. In other words, $d_2^\lambda \gg (\lambda/\lambda_*)d_2^{\lambda_*}+(1-\lambda/\lambda_*)a_2$. If there are more than one τ_{2,c_1^λ}-minimal sets strongly above a_2, Theorem 3.13 in [28] ensures that $\mu d_2^\lambda+(1-\mu)\tilde a_2$ is τ_{2,c_1^λ}-minimal for $\mu\in(0,1]$, which together with Proposition 3.5(ii) (applied to the labeling w.r.t. K^{λ_*}) ensures that $\{c_1^\lambda,\mu d_2^\lambda+(1-\mu)\tilde a_2\}$ is minimal for $\mu<1$ close enough to 1. That is, there are infinitely many elements in $\mathcal{M}_{a,K^{\lambda_*}}$ labeled on their first component w.r.t. K^{λ_*} by λ/λ_*, impossible. Now assume that $\lambda_+=\sup J$ is finite and that there exists a $\tau_{2,c_1^{\lambda_+}}$-minimal set strongly above $d_2^{\lambda_+}$. Again Theorem 3.13 in [28] ensures that $\mu d_2^{\lambda_+}+(1-\mu)\tilde a_2$ is $\tau_{2,c_1^{\lambda_+}}$-minimal for $\mu\in(0,\mu_1]$ with $\mu_1>1$. In addition,

$$\big(c_1^{\lambda_+}\big)'(\omega)=F_1\big(\omega,c_1^{\lambda_+}(\omega),d_2^{\lambda_+}(\omega)\big)\leqslant F_1\big(\omega,c_1^{\lambda_+}(\omega),c_2^{\lambda_+}(\omega)\big)\leqslant\big(c_1^{\lambda_+}\big)'(\omega),$$

so that $(c_1^{\lambda_+})'(\omega)=F_1(\omega,c_1^{\lambda_+}(\omega),\mu d_2^{\lambda_+}(\omega)+(1-\mu)\tilde a_2(\omega))$ for $\mu\in[1,\mu_1]$. Consequently there exist minimal sets above K^{λ_+}, impossible.

Let us now deduce that the (increasing and continuous) map $J\to X_2$, $\lambda\mapsto d_2^\lambda(\omega)$ is concave for any $\omega\in\Omega$. We fix $\lambda_1,\lambda_2\in J$ with $\lambda_1<\lambda_2$. Since by (2.5) any convex combination of two equilibria is a subequilibrium, Proposition 2.9 implies that, for any $\mu\in(0,1)$, the map $\tilde d_2^\lambda=\mu d_2^{\lambda_1}+(1-\mu)d_2^{\lambda_2}$, which is strongly above a_2, is a continuous subequilibrium for τ_{2,c_1^λ}, with $\lambda=\mu\lambda_1+(1-\mu)\lambda_2$. Hence Proposition 3.6(iii) in [28] ensures that its omega-limit set is a τ_{2,c_1^λ}-minimal set above $\tilde d_2^\lambda$, which necessarily agrees with d_2^λ. In other words, $d_2^{\mu\lambda_1+(1-\mu)\lambda_2}\geqslant \mu d_2^{\lambda_1}+(1-\mu)d_2^{\lambda_2}$, as asserted.

Take finally $\lambda_1\in J$. If there exists $\lambda_2\in J-(0,\lambda_1]$ with $d_2^{\lambda_1}=d_2^{\lambda_2}$, the concavity and increasing character of $\lambda\mapsto d_2^\lambda$ ensure that $d_2^\lambda=d_2^{\lambda_1}$ for any $\lambda\in J-(0,\lambda_1]$. If, on the contrary, $d_2^{\lambda_1}<d_2^{\lambda_2}$ for any $\lambda_2\in J-(0,\lambda_1]$, Proposition 2.4 ensures that $d_2^{\lambda_1}\ll d_2^{\lambda_2}$, and this completes the proof of the last assertion. □

Remarks 3.10. (1) Note that in situation (b) of the previous theorem, if we take $\lambda_0>1$ in $J-J_1$, any K^λ for $\lambda\in J$ is labeled w.r.t. K^{λ_0} only on its first component. We rename K^{λ_0} as K^1. The arguments of Proposition 3.5(ii) combined with the fact that c^λ is a super-equilibrium if $\lambda>1$ prove that $(c_1^\lambda)'(\omega)=F_1(\omega,c_1^\lambda(\omega),x_2)$ for $\lambda\in J$ and $x_2\gg\min(c_2^\lambda(\omega),d_2^\lambda(\omega))$ for ODEs, and the analogous equalities in the delay and parabolic cases. Taking limit as $\lambda\to 0^+$, we conclude that $a_1'(\omega)=F_1(\omega,a_1(\omega),x_2)$ whenever $x_2\gg a_2(\omega)$, and this is independent of the choice of the initial λ_0. Symmetric properties hold in case (c).

(2) Let us take $\lambda_1<\lambda_2$ in J and $\mu\in(0,1)$, and call $\lambda=\mu\lambda_1+(1-\mu)\lambda_2$. As stated in the previous result, the continuous map $\tilde d_2^\lambda=\mu d_2^{\lambda_1}+(1-\mu)d_2^{\lambda_2}$ is below d_2^λ. The first argument in the proof of Proposition 3.2(ii) shows that either $\tilde d_2^\lambda=d_2^\lambda$ or $\tilde d_2^\lambda\ll d_2^\lambda$. This property will be important in the proofs of the results of Section 4.

(3) Assume that the dynamics fits case (b). Then the map sending $\lambda\in(0,1]$ to $K_\omega^\lambda\subset X$ is continuous, increasing and concave for any $\omega\in\Omega$, since $\lambda\to(c_1^\lambda(\omega),d_2^\lambda(\omega))$ satisfies these properties. Symmetric conclusions hold in case (c). In other words, and roughly speaking, Theorem 3.9 says that, for each $\omega\in\Omega$, the section over $\{\omega\}\times X$ of the elements of $\mathcal{M}_a$, considered as a map of the first coordinate, is the graph of a concave or convex map which can be a straight line (for instance, if (a) holds). This is the exact situation in the autonomous case.

(4) Note that Theorems 3.6 and 3.9 ensure that the existence of a semicontinuous subequilibrium $a\ll K^1$ with a_1 and a_2 not continuous is only possible in Case A1 of Theorem 3.8 in [28], and with a_1 continuous but a_2 discontinuous (or viceversa) is only possible in cases (a) and (b) (or (a) and (c)) of Theorem 3.9.

Next theorem describes the different possibilities for the set $\mathcal{M}$. To understand its statements is important to have in mind that we cannot assert that a minimal set M is a copy of the base unless we previously know that it is strongly above a semicontinuous subequilibrium. However, we will see that at least one of its fiber components satisfies this condition, and we can determine the area in which M can lie depending on the shape of $\mathcal{M}_a$.

Let us fix some additional notation. Assume that for a given minimal set M there is $\lambda_M \in \mathbb{R}$ such that $x_1 = c_1^{\lambda_M}(\omega)$ for any $(\omega, x_1, x_2) \in M$. In this case the set $M_2 = \{(\omega, x_2) \mid (\omega, c_1^{\lambda_M}(\omega), x_2) \in M\}$ is minimal for the marginal semiflow $\tau_{2,c_1^{\lambda_M}}$. We represent this situation as $M = \{c_1^{\lambda_M}, M_2\}$ and say that $c_1^{\lambda_M}$ is the first component of M. We write $M_2 = \{d_2\}$ in order to say that M_2 is the graph of the continuous map d_2. Analogously, we write $M = \{M_1, c_2^{\lambda_M}\}$ and say that $c_2^{\lambda_M}$ is the second component of M if $x_2 = c_2^{\lambda_M}(\omega)$ for any $(\omega, x_1, x_2) \in M$. Finally, we also define $\mathcal{M}_a^* = \mathcal{M}_a \cup \{K^0\}$, where K^0 is the minimal set above a and outside $\mathcal{M}_a$ appearing in the next consequence of Theorem 3.9.

Corollary 3.11. *Under the hypotheses of Theorem* 3.9, *assume that* (a) *holds and a_1 is C^1 or that* (b) *holds, write $K^\lambda = \{c_1^\lambda, d_2^\lambda\}$, and define $d_2^0 = \lim_{\lambda\to 0^+} d_2^\lambda$. Then there is a minimal set $K^0 = \{a_1, K_2^0\}$ such that*

- *either $K_0^2 = \{d_0^2\} \gg a_2$,*
- *or $a_2 \leqslant K_2^0 \leqslant d_2^0$, $(K_2^0)_\omega = \{a_2(\omega)\} = \{d_2^0(\omega)\}$ in a residual subset of Ω, and $x_2 \ll d_2^0(\omega)$ if $x_2 \neq d_2^0(\omega)$ and $x_2 \in (K_2^0)_\omega$.*

In addition, either $K^0 \ll K^\lambda \ll K^{\lambda_0}$ for a $\lambda_0 \in J$ and any $\lambda \in (0, \lambda_0)$, or $K^0 = \{a_1, c_2^1\}$ and $K^\lambda = \{c_1^\lambda, c_2^1\}$ for any $\lambda \in J$. And, if (b) *holds, K_2^0 is the top minimal set for the semiflow τ_{2,a_1}. Symmetric properties hold in situations* (a) *if a_2 is C^1 or* (c)*.*

Proof. Recall that a_1 is C^1 also if (b) holds. Since the family of functions (d_2^λ) decreases as $\lambda \downarrow 0$, they define a limit $d_2^0 : \Omega \to X_2$. It is immediate to check that (a_1, d_2^0) is an equilibrium above a. In addition, the set $C_d = \{(\omega, x_1, x_2) \in \Omega \times X \mid (x_1, x_2) \leqslant (a_1(\omega), d_2^0(\omega))\}$ is a closed subset of $\Omega \times X$ (see [2]); and the set $\Gamma_d = \text{closure}_{\Omega\times X}\{(\omega, a_1(\omega), d_2^0(\omega)) \mid \omega \in \Omega\}$ is a compact subset of $\Omega \times X$. This result is trivial when $X_2 = \mathbb{R}$, and follows from the fact that $\tau_t(M)$ is relatively compact for any bounded set M for $t > 1$ in the delay case and for $t > 0$ in the parabolic case, since $\bigcup_{\lambda\in(0,1]} K^\lambda = \tau_t(\bigcup_{\lambda\in(0,1]} K^\lambda)$. Let $R = \{\omega \in \Omega \mid a_2$ and d_2^0 are continuous at $\omega\}$, a residual set. According to Theorem 3.6 in [21], the omega-limit set of $(\omega_0, a_1(\omega_0), d_2^0(\omega_0))$ for a point $\omega_0 \in R$ contains a minimal set K^0 given by an almost automorphic extension of the base, with $(a_1(\omega), a_2(\omega)) \leqslant (x_1, x_2) \leqslant (a_1(\omega), d_2^0(\omega))$ for any $(\omega, x_1, x_2) \in K^0$, and with $(K^0)_\omega = \{(a_1(\omega), d_2^0(\omega))\}$ for $\omega \in R$. Therefore, $K^0 = \{a_1, K_2^0\}$, with $(K_2^0)_\omega = \{d_2^0(\omega)\}$ if $\omega \in R$. In addition, K_2^0 is a minimal set for the marginal semiflow τ_{2,a_1}, for which d_2^0 is an equilibrium and a_2 a subequilibrium.

Assume first that a_2 and d_2^0 do not agree in R, which ensures the existence of $\omega_0 \in R$ with $a_2(\omega_0) \ll d_2^0(\omega_0)$. Let us deduce that $a_2 \ll K_2^0$. We fix $(\omega, x) \in K_2^0$ and a τ_{2,a_1}-backward orbit $\{(\omega\cdot s, x_s) \mid s \leqslant 0\} \subset K_2^0$, and choose $(s_n) \downarrow -\infty$ with $\lim_{n\to\infty} \omega\cdot s_n = \omega_0$. Then, since $\lim_{n\to\infty}(x_{s_n} - a_2(\omega\cdot s_n)) = d_2^0(\omega_0) - a_2(\omega_0) \gg 0$, there exists s_n with $x_{s_n} - a_2(\omega\cdot s_n) \gg 0$. Proposition 3.4 in [28] ensures that $x = u_2^{a_1}(-s_n, \omega\cdot s_n, x_{s_n}) \gg u_2^{a_1}(-s_n, \omega\cdot s_n, a_2(\omega\cdot s_n)) \geqslant a_2(\omega)$, as asserted. This means that τ_{2,a_1} satisfies the hypotheses of Proposition 3.6(vi) in [28], and this result ensures that K_0^2 is a uniformly stable copy of the base: $K_0^2 = \{\tilde{d}_2^0\}$. In addition, d_2^0 and $\tilde{d}_2^0$ agree in R, which together with the uniform stability ensure that they agree in the whole Ω. That is, $K^0 = \{a_1, d_2^0\}$ is a minimal set, which agrees with the first possibility of the statement.

Let us now consider the case that a_2 and d_2^0 agree in R. If $x_2 < d_2^0(\omega)$ for an $x_2 \in (K_2^0)_\omega$ (with $\omega \notin R$) then $u_2(-1, \omega, a_1(\omega), x_2) < d_2^0(\omega\cdot(-1))$ for any backward orbit inside K^0, so that Proposition 2.4 ensures that $x_2 \ll d_2^0(\omega)$. This means that in this case the second possibility of the statement for K^0 holds.

It is obvious that if $d_2^\lambda = c_2^1$ for any λ, then $d_2^0 = c_2^1$ and $K^0 = \{a_1, c_2^1\}$. If this is not the case, and according to the last point in the description of case (b), there exists $\lambda_0 \in J$ such that the map $(0, \lambda_0) \to X_2$, $\lambda \mapsto d^\lambda(\omega)$ is strongly increasing for any $\omega \in \Omega$, and hence $K_2^0 \leqslant d_2^0 \ll d_2^\lambda$ for any $\lambda \in J$.

Finally, assume that (b) holds and, by contradiction, the existence of a τ_{2,a_1}-minimal set $M_2 > K_2^0$. Let us check that $M_2 \gg a_2$. This property is obvious if $K_2^0 = \{d_2^0\} \gg a_2$, so that assume that a_2 and d_2^0 agree in R. Given $(\omega, x_2) \in M_2$ we can find $(\omega \cdot (-t), y_2) \in M_2$ with $u_2^{a_1}(t, \omega \cdot (-t), y_2) = (\omega, x_2)$ and $y_2 > a_2(\omega \cdot (-t))$: otherwise we would deduce from the minimality of the base flow the existence of $(\omega_0, a_2(\omega_0)) \in M_2$ with $\omega_0 \in R$, impossible. Proposition 2.4 ensures that $x_2 \gg a_2(\omega)$, as asserted. This fact and the arguments of Proposition 2.9 guarantee that the semiflow τ_{2,a_1} is either in Case A1 of [28] or in the special Case A2 described by Theorem 3.13 in [28]. In any case, M_2 is a copy of the base, $M_2 = \{m_2\}$, with $m_2 \gg K_2^0 \geqslant a_2$ and $m_2(\omega_0) \gg d_2^0(\omega_0)$ if $\omega_0 \in R$. Remark 3.10(1) ensures that $\{a_1\}$ is τ_{1,m_2}-minimal, so that $M = \{a_1, m_2\}$ is τ-minimal. Let us now fix $\lambda_1 \in J$ with $M \leqslant K^{\lambda_1}$, whose existence is ensured by the monotonicity of the semiflow and the definition of c^λ, and write $K^{\lambda_1} = \{c_1^{\lambda_1}, d_2^{\lambda_1}\}$. Then $(c_1^{\rho\lambda_1}, \rho d_2^{\lambda_1} + (1-\rho) m_2)$ is a continuous subequilibrium for any $\rho \in (0,1)$, which implies that its second component is a $\tau_{2,c_1^{\rho\lambda_1}}$-subequilibrium. According to Theorem 3.6 in [21], this subequilibrium determines a minimal set above it. Let us fix $\omega_0 \in R$ and note that $\rho d_2^{\lambda_1}(\omega_0) + (1-\rho) m_2(\omega_0) \gg d_2^{\rho\lambda_1}(\omega_0)$ for $\rho > 0$ small enough. This contradicts the fact that $d_2^{\rho\lambda_1}$ is the top $\tau_{2,c_1^{\rho\lambda_1}}$-minimal set, guaranteed by the description of case (b) in Theorem 3.9, and hence completes the proof. $\square$

Theorem 3.12. *Assume that the family* (2.2), (2.3) *or* (2.4) *satisfies* (H1)–(H3), *and that the dynamics above a fits Case* A2 *of Theorem* 3.8 *in* [28]. *Assume also the existence of a reference minimal set* $K^1 = \{c^1\} = \{c_1^1, c_2^1\} \gg a$ *for which there exists a single label. Then there exists a closed interval* $J_* \supseteq J \cup \{0\}$ *with* $\sup J_* = \sup J$ *such that the set* $\mathcal{M}$ *is in one of the following situations*:

($\tilde{\text{a}}$) $\mathcal{M} = \{\{\tilde{c}^\lambda\} \mid \lambda \in J_*\}$.

($\tilde{\text{b}}$) $\mathcal{M} \supseteq \{M^\lambda \mid \lambda \in J_*\}$ *for a continuous and strictly increasing family* $(M^\lambda)_{\lambda \in J_*}$ *of minimal sets with* $M^\lambda = \{\tilde{c}_1^\lambda, d_2^\lambda\}$ *for* $\lambda > \inf J_*$, *where the map* $J_* - \{\inf J_*\} \to X_2$, $\lambda \mapsto d_2^\lambda(\omega)$ *is concave for every* $\omega \in \Omega$, *and either there is* $\lambda \in J_* - \{\inf J_*\}$ *with* $d_2^\lambda \neq \tilde{c}_2^\lambda$ *or* $\mathcal{M} - \{M^\lambda \mid \lambda \in J_*\}$ *is nonempty. In the case that* $\lambda_- = \inf J_* > -\infty$, *the map* $d_2^{\lambda_-} = \lim_{\lambda \to (\lambda_-)^+} d_2^\lambda$ *defines a semicontinuous equilibrium* $(\tilde{c}_1^{\lambda_-}, d_2^{\lambda_-})$ *giving rise to the minimal set* $M^{\lambda_-} = \{\tilde{c}_1^{\lambda_-}, D_2^{\lambda_-}\} \leqslant (\tilde{c}_1^{\lambda_-}, d_2^{\lambda_-})$, *with* $(D_2^{\lambda_-})_\omega = \{d_2^{\lambda_-}(\omega)\}$ *in a residual subset of* Ω, *and with* $x_2 \ll d_2^{\lambda_-}(\omega)$ *if they are different and* $x_2 \in (D_2^{\lambda_-})_\omega$. *In addition,* (i) $M^\lambda = K^\lambda$ *for every* $\lambda \geqslant 0$; (ii) *any* $M \in \mathcal{M} - \{M^\lambda \mid \lambda \in J_*\}$, *if it exists, is of the form* $\{\tilde{c}_1^\lambda, M_2^\lambda\}$ *for some* $\lambda \in J_*$, *with* $M_2^\lambda \ll d_2^\lambda$ *if* $\lambda > \inf J_*$ *or* $M_2^{\lambda_-} \ll D_2^{\lambda_-}$ *if* $\lambda = \lambda_- = \inf J_*$; *and* (iii) *if there are two minimal sets* $M = \{\tilde{c}_1^{\lambda_1}, M_2\} \neq M^{\lambda_1}$ *and* $N = \{\tilde{c}_1^{\lambda_2}, N_2\} \neq M^{\lambda_2}$, *then* $M_2 \not< N_2$ *in the following cases*: *if* $\lambda_1 < \lambda_2$, *if* $\lambda_1 = \lambda_2 \in \operatorname{Int} J_* \cap (0, \infty)$, *or if* $\lambda_1 = \lambda_2 \in \operatorname{Int} J_*$ *and in addition there is* $\lambda_0 \in J_*$ *with* $M^{\lambda_0} \ll M^{\lambda_1}$.

($\tilde{\text{c}}$) *The symmetric situation to* ($\tilde{\text{b}}$) *occurs,* c_2^λ *being the second component of any* $M \in \mathcal{M}$ *for some* $\lambda \in J_*$.

Proof. Theorem 3.9 and Corollary 3.11 provide the whole description if $\mathcal{M} = \mathcal{M}_a^*$, so that we work in the case that $\mathcal{M} - \mathcal{M}_a^*$ is nonempty.

Assume that any minimal set is of the form $\{\tilde{c}^\lambda\}$, and hence that the minimal set K^0 provided by Corollary 3.11 agrees with $\{\tilde{a}\}$. Then the set $J_* = \{\lambda \in \mathbb{R} \mid \{\tilde{c}^\lambda\}$ is minimal$\}$ satisfies $J_* \cap [0, \infty) = J \cup \{0\}$. In addition, if $\lambda < 0$ belongs to J_*, Theorem 3.9 applied to the (sub)equilibrium $\tilde{c}^\lambda$ ensures that $[\lambda, 0] \subset J_*$, so that it is an interval. And trivially it is closed: case ($\tilde{\text{a}}$) holds.

In what follows, and until otherwise indicated, we will work in the case that $\mathcal{M}_a$ fits case (b) of Theorem 3.9, which obviously precludes cases ($\tilde{\text{a}}$) and ($\tilde{\text{c}}$) of the statement. Our purpose is hence to prove all the assertions in ($\tilde{\text{b}}$). Recall that in this case $\tilde{c}_1^\lambda = c_1^\lambda$ for any $\lambda \geqslant 0$ and, as said in Remark 3.10(1), by changing the reference set K^1 if needed, we get any element of $\mathcal{M}_a$ labeled w.r.t. K^1 only on its first component, and this makes no difference in the statements of the theorem. We do so and write $K^\lambda = \{c_1^\lambda, d_2^\lambda\}$ for $\lambda \in J$. We begin by assuming the existence of $M \in \mathcal{M} - \mathcal{M}_a^*$ with

$M \leqslant K^\lambda$ for every $\lambda \in J$, which in particular implies $M < K^0$, and we make an extra and fundamental assumption, which will be removed later: that M is an equilibrium, say $M = \{m\} = \{m_1, m_2\}$. So that $m_1 \leqslant a_1$ and there are two possibilities: either $m_2 \ll d_2^\lambda$ for any $\lambda \in J$, or $m_2 = c_2^1$.

Let us analyze the first one. We take m as the new initial subequilibrium and label the elements of the set $\mathcal{M}_m$ in terms of m and K^1. Assume first that all of them are labeled on their first component. (In fact we will check later that this one is the unique possible option.) This means that $c_1^\lambda = \mu_\lambda c_1^1 + (1-\mu_\lambda) m_1$ for a certain μ_λ, so that there exists $\lambda_M < 0$ such that $m_1 = c_1^{\lambda_M}$. In addition, the multiple-labeling situation is precluded, since it would imply a multiple-labeling situation also for $\mathcal{M}_a$. According to Theorem 3.9(b) and Corollary 3.11, there is a continuous and strictly increasing family $(M^\lambda)_{\lambda \in [\lambda_M, 0] \cup J}$ of minimal sets above M, with $M^\lambda = \{c_1^\lambda, d_2^\lambda\}$ for $\lambda > \lambda_M$, with $M^{\lambda_M} = \{c_1^{\lambda_M}, D_2^{\lambda_M}\}$ where the relation between $D_2^{\lambda_M}$ and $d_2^{\lambda_M} = \lim_{\lambda \to (\lambda_M)^+} d_2^\lambda$ is the same as between K^0 and d_2^0 in Corollary 3.11, and with $M^\lambda = K^\lambda$ if $\lambda \geqslant 0$; and the map $[\lambda_M, 0] \cup J \to X_2$, $\lambda \mapsto d_2^\lambda(\omega)$ is concave for every $\omega \in \Omega$. Now assume by contradiction that the elements of $\mathcal{M}_m$ are labeled only on their second component, so that it fits case (c) of Theorem 3.9. Note that for $\lambda \in (0,1)$ we have $d_2^\lambda = \rho_\lambda c_2^1 + (1-\rho_\lambda) m_2$ with $c_1^\lambda \gg \rho_\lambda c_1^1 + (1-\rho_\lambda) m_1$, and that ρ_λ (and hence d_2^λ) increases strongly with λ in $(0,1)$. Proposition 3.5(i) ensures that $(d_2^\lambda)'(\omega) = F_2(\omega, \rho_\lambda c_1^1(\omega) + (1-\rho_\lambda) m_1(\omega), d_2^\lambda(\omega)) \leqslant F_2(\omega, c_1^\lambda(\omega), d_2^\lambda(\omega)) = (d_2^\lambda)'(\omega)$. We choose $\mu < \lambda$ in $(0,1)$ with $\rho_\lambda c_1^1 + (1-\rho_\lambda) m_1 \ll c_1^\mu$ in order to conclude that $\{d_2^\lambda\}$ is a τ_{2,c_1^μ} minimal set, contradicting the description of the case (b) for $\mathcal{M}_a$ provided by Theorem 3.9. Note that, although M and M^{λ_M} share the first component, it can be $M < M^{\lambda_M}$. Proposition 2.4 ensures that, in this case, $M_2 \ll D_2^{\lambda_M}$ (where possibly $D_2^{\lambda_M} = \{d_2^{\lambda_M}\}$).

If, on the contrary, $m_2 = c_2^1$, then $d_2^\lambda = c_2^1$ for any $\lambda \in [0,1]$, and hence $\{m_1\}$ as well as $\{c_1^\lambda\}$ for $\lambda \in [0,1]$ are τ_{1,c_2^1}-minimal sets. According to Theorem 3.13 in [28], $m_1 = c_1^{\lambda_M}$ for a $\lambda_M \leqslant 0$, and any $\{c_1^\lambda\}$ with $\lambda \in [\lambda_M, 1]$ is τ_{1,c_2^1}-minimal. Let us check that $\{c_2^1\}$ is τ_{2,c_1^λ}-minimal for any $\lambda \in [\lambda_M, 1]$. This follows as in previous cases from $(c_2^1)'(\omega) = F_2(\omega, c_1^{\lambda_M}(\omega), c_2^1(\omega)) \leqslant F_2(\omega, c_1^1(\omega), c_2^1(\omega)) = (c_2^1)'(\omega)$. This and the last assertion in Theorem 3.9(b) ensure that $\{\{c_1^\lambda, c_2^1\} \mid \lambda \in [\lambda_M, 0] \cup J\} \subset \mathcal{M}$. To unify notation we represent $M^\lambda = \{c_1^\lambda, d_2^\lambda\}$ with $d_2^\lambda = c_2^1$. The family $(M^\lambda)_{\lambda \in [\lambda_M, 0] \cup J}$ satisfies in this case the same properties as in the previous one, and $M = M^{\lambda_M}$.

Let us now remove the hypothesis that M is an equilibrium. As in the proof of Theorem 3.7, we consider the semiflow τ^M defined in the product space $M \times X$ by trivial extensions of the functions F_i. Recall that $M_* = \{(\omega, z, z) \mid (\omega, z) \in M\}$ is a copy of the base for τ^M: $M_* = \{m^*\}$, with $m^*(\omega, z) = z$. It is also clear that any τ^M-minimal set is contained in a set $\{(\omega, z, x) \mid (\omega, z) \in M \text{ and } (\omega, x) \in K\}$ for a τ-minimal set K. This implies that the τ^M-minimal sets strongly above the subequilibrium a^* given by $a^*(\omega, z) = a(\omega)$ are exactly $K_*^\lambda = \{(\omega, z, c_1^\lambda(\omega), d_2^\lambda(\omega)) \mid (\omega, z) \in M\}$, for $\lambda \in J$, since these sets are τ^M-minimal. The previously proved properties ensure that $M_* = \{(\omega, z, c_1^{\lambda_M}(\omega), m_2^*(\omega, z)) \mid (\omega, z) \in M\}$ for a $\lambda_M \leqslant 0$, as well as the existence of a continuous and strictly increasing family $(M_*^\lambda)_{\lambda \in J_M}$ of τ^M-minimal sets above M_* for $J_M = [\lambda_M, 0] \cup J$, with $M_*^\lambda = \{(\omega, z, c_1^\lambda(\omega), (d_2^\lambda)^*(\omega, z)) \mid (\omega, z) \in M\}$ for $\lambda > \lambda_M$, with $M_*^{\lambda_M} = \{c_1^{\lambda_M}, (D_2^{\lambda_M})_*\}$ where $(D_2^{\lambda_M})_*$ is associated to $(d_2^{\lambda_M})^* = \lim_{\lambda \to (\lambda_M)^+} (d_2^\lambda)^*$, and with $M^\lambda = K^\lambda$ if $\lambda \geqslant 0$. Consequently, $M = \{c_1^{\lambda_M}, M_2\}$ and the sets $M^\lambda = \{(\omega, c_1^\lambda(\omega), (d_2^\lambda)^*(\omega, z)) \mid (\omega, z) \in M\} = \{c_1^\lambda, D_2^\lambda\}$ for any $\lambda \in J_M - \{\lambda_M\}$ and $M^{\lambda_M} = \{(\omega, c_1^{\lambda_M}(\omega), x_2) \mid \exists z \text{ with } (\omega, z, x_2) \in (D_2^{\lambda_M})_*\} = \{c_1^\lambda, D_2^{\lambda_M}\}$ are τ-minimal. Let us check that M^λ is in fact a copy of the base if $\lambda > \lambda_M$. As seen above, if $D_2^\lambda = M_2$ then $D_2^\lambda = \{d_2^\lambda\}$ with $d_2^\lambda = c_2^1$ for any $\lambda \in J_M$. Otherwise, $M^\lambda \gg M$, and hence the assertion follows from Theorem 3.1: $D_2^\lambda = \{d_2^\lambda\}$. As before, the map $[\lambda_M, 0] \cup J \to X_2$, $\lambda \mapsto d_2^\lambda(\omega)$ is concave for every $\omega \in \Omega$.

Our next purpose is to check that, for $\lambda \in J_M$, D_2^λ is the top minimal set for the semiflow τ_{2,c_1^λ}, property which we will use in the next paragraph to show that J_* is well defined. It follows directly from Theorem 3.9 (applied to $\mathcal{M}_m$) and Corollary 3.11 in the first case analyzed, and also in the second one if $\lambda \geqslant 0$. In the remaining case to be considered, $d_2^\lambda = c_2^1$ for any $\lambda \in J_M$. We fix $\lambda_0 \in [\lambda_M, 0)$. The concavity on $\mathbb{R}$ of the map $h_1(\lambda) = F_1(\omega, c_1^\lambda(\omega), x_2) - (c_1^\lambda)'(\omega)$ for $\omega \in \Omega$, $\lambda \leqslant 1$ and $x_2 \gg c_2^1(\omega)$ fixed and the fact that it vanishes in J (in turn ensured by Remark 3.10(1)) ensure that

$(c_1^{\lambda_0})'(\omega) \geqslant F_1(\omega, c_1^{\lambda_0}(\omega), x_2)$; this together with the equality $(c_1^{\lambda_0})'(\omega) = F_1(\omega, c_1^{\lambda_0}(\omega), c_2^1(\omega))$ and the increasing properties of F_1 provide $(c_1^{\lambda_0})'(\omega) = F_1(\omega, c_1^{\lambda_0}(\omega), x_2)$, and the equality holds for every $\omega \in \Omega$ whenever $x_2 \geqslant d_2^{\lambda_0}(\omega) = c_2^1(\omega)$. It is easy to deduce that if $N_2 > c_2^1$ is a $\tau_{2,c_1^{\lambda_0}}$-minimal set then $\{c_1^{\lambda_0}, N_2\} = \{(\omega, c_1^{\lambda_0}(\omega), x_2) \mid (\omega, x_2) \in N_2\}$ is minimal. But it is not below any K^λ, impossible.

Assume that a new $N \in \mathcal{M} - \mathcal{M}_a$ exists with $N \leqslant K^\lambda$ for every $\lambda \geqslant 0$. Since D_2^λ is the top τ_{2,c_1^λ}-minimal set, the map $J_M \cup J_N \to \mathcal{P}_c(\Omega \times X_2), \lambda \mapsto D_2^\lambda$ is well defined and satisfies the previously proved properties. So that we define J_* as the union of all the intervals J_M corresponding to minimal sets M with $M \leqslant K^\lambda$ for every $\lambda \in J$. If no minimal set with this property exists, we define $J_* = J \cup \{0\}$. The proof of the stated properties of the map $d_2^{\lambda_-}$ and the minimal set M^{λ_-} in the case that $\lambda_- = \inf J_* > -\infty$ is made in Corollary 3.11.

Let us now check (ii) and (iii) of ($\tilde{\text{b}}$). We take $M \in \mathcal{M} - \mathcal{M}_a^*$ with $M \neq M^\lambda$ for every $\lambda \in J_*$ and define λ_* as the minimum value of $\lambda \in J_*$ with $M \leqslant M^\lambda$. There is nothing to prove if $\lambda_* \leqslant 0$, since in this case M is one of the minimal sets analyzed above. So that assume that $\lambda_* > 0$, so that $M^{\lambda_*} = K^{\lambda_*} = \{c_1^{\lambda_*}, d_2^{\lambda_*}\}$. The arguments of Proposition 3.2(ii) show that either $M = \{c_1^{\lambda_*}, M_2\}$ with $M_2 \ll d_2^{\lambda_*}$, as we assert, or $M = \{M_1, d_2^{\lambda_*}\}$ with $M_1 \ll c_1^{\lambda_*}$. Let us check that this second option is impossible. Since $(c_1^\lambda)'(\omega) = F_1(\omega, c_1^\lambda(\omega), d_2^{\lambda_*}(\omega))$ for $\lambda \in (0, \lambda_*]$ (see Remark 3.10(1)), all the sets M_1 and $(c_1^\lambda)_{\lambda \in (0,\lambda_*]}$ are $\tau_{1,d_2^{\lambda_*}}$-minimal, so that Theorem 3.13 in [28] ensures that $M_1 = \{c_1^{\lambda_1}\}$ for a $\lambda_1 < \lambda_*$ and hence $M = \{c_1^{\lambda_1}, d_2^{\lambda_*}\}$. Since, in addition, $(d_2^{\lambda_*})'(\omega) = F_2(\omega, c_1^{\lambda_1}(\omega), d_2^{\lambda_*}(\omega)) = F_2(\omega, c_1^{\lambda_*}(\omega), d_2^{\lambda_*}(\omega))$, we conclude that $\{c_1^\lambda, d_2^{\lambda_*}\}$ is minimal for $\lambda \in [\lambda_1, \lambda_*]$. In turn this implies that $M^\lambda = \{c_1^\lambda, d_2^{\lambda_*}\}$ if $\lambda_1 \leqslant \lambda \in J_*$, so that $M = M^{\lambda_1}$. This completes the proof of (ii). To prove (iii), assume the existence of such sets M and N for $\lambda_1 \leqslant \lambda_2$, with $M_2 < N_2$. Proposition 2.4 ensures that $M_2 \ll N_2$. In the case $\lambda_1 < \lambda_2$, $M \ll N$, so that as seen above N has to be one of the sets M^λ. Let us now consider the case $\lambda_1 = \lambda_2 \in \operatorname{Int} J_*$. We can assume without loss of generality that $\lambda_1 < 1$, by changing the reference minimal set if needed. In the two remaining situations considered in (iii), there exists a subequilibrium strongly below M^{λ_1}: either a or the equilibrium $(\tilde{c}_1^{\lambda_2}, d_2^{\lambda_2})$ for a $\lambda_2 \in (\lambda_0, \lambda_1)$. Theorem 3.9 ensures that $d_2^{\lambda_1}$ is the unique $\tau_{2,c_1^{\lambda_1}}$-minimal set strongly above the second component of the subequilibrium. But the existence of the strongly ordered $\tau_{2,c_1^{\lambda_1}}$-minimal sets M_2, N_2 and $D_2^{\lambda_1}$ allows us to apply Theorem 3.13 in [28] (by extending the semiflow to the new base $\Omega \times M$) in order to deduce the existence of $\tau_{2,c_1^{\lambda_1}}$-copies of the base as close to $d_2^{\lambda_1}$ as we wish, which contradicts the previous property. (We point out that, although Theorem 3.13 in [28] is stated and proved for a flow in the base, it also holds in this base semiflow setting, since the condition (h4) there required is now satisfied for any element in the base.)

Obviously, if $\mathcal{M}_a$ fits case (c) of Theorem 3.9, then $\mathcal{M}$ fits case ($\tilde{\text{c}}$).

To complete the proof of the theorem, we will check that if $\mathcal{M}_a$ fits case (a) of Theorem 3.9 and $\mathcal{M}$ does not fit ($\tilde{\text{a}}$) above, then it fits ($\tilde{\text{b}}$) or ($\tilde{\text{c}}$). So that we assume the existence of $M \in \mathcal{M} - \mathcal{M}_a^*$ with $M \neq \{\tilde{c}^\lambda\}$ for any $\lambda \in \mathbb{R}$. Assume first that $M = \{m\} \leqslant K^\lambda$ for any $\lambda > 0$, and hence $m \ll K^\lambda$ for any $\lambda \geqslant 0$. Then, if all the elements K^λ of $\mathcal{M}_m$ are labeled on their first (resp. second) component w.r.t. m and K^1, we obtain $M = \{\tilde{c}_1^{\lambda_M}, m_2\}$ with $m_2 \ll \tilde{c}_2^{\lambda_M}$ (resp. $M = \{m_1, \tilde{c}_2^{\lambda_M}\}$ with $m_1 \ll \tilde{c}_1^{\lambda_M}$), and all the previous arguments can be repeated with a replaced by m to show that ($\tilde{\text{b}}$) (resp. ($\tilde{\text{c}}$)) holds. If, on the contrary, there is a minimum $\lambda_1 > 0$ such that $M = \{m\} \leqslant K^{\lambda_1}$, then either $M = \{\tilde{c}_1^{\lambda_1}, m_2\}$ with $m_2 \ll \tilde{c}_2^{\lambda_1}$ or $M = \{m_1, \tilde{c}_2^{\lambda_1}\}$ with $m_1 \ll \tilde{c}_1^{\lambda_1}$. If $\lambda_1 < \sup J$ and the first (resp. second) equality holds, then the minimal sets of $\mathcal{M}_m$ are in situation (b) (resp. (c)) w.r.t. m and K^1, so that ($\tilde{\text{b}}$) (resp. ($\tilde{\text{c}}$)) holds. Note that in these cases we cannot assert that a_1 (resp. a_2) is C^1, although we know that at least one of the components of a is. To remove the assumption that M is an equilibrium, we work as usually with the semiflow τ^M to obtain a family of τ^M-minimal sets $(M_*^\lambda)_{\lambda \in J_*}$ having $\tilde{c}_1^\lambda$ as first component and projecting onto the minimal sets $(M^\lambda)_{\lambda \in J_*}$, which satisfy the stated properties. (To prove that M^λ is a copy if the base if $\lambda > \inf J_*$ we apply the results of [10] and [23] as before.) Finally, let us check that if $\lambda_+ = \sup J < \infty$, the existence of a minimal set $M = \{\tilde{c}_1^{\lambda_+}, M_2\}$ with

$M_2 \ll \tilde{c}_2^{\lambda_+}$ precludes the existence of $N = \{N_1, \tilde{c}_2^{\lambda_+}\}$ with $N_1 \ll \tilde{c}_1^{\lambda_+}$. Assuming the existence of such a minimal set, we fix $\lambda_1 \in (0, \lambda_+)$ with $M_2 \ll c_2^{\lambda_1}$, reason as several times above in order to show that $\tilde{c}_1^{\lambda_+}$ is $\tau_{1,\tilde{c}_2^\lambda}$-minimal for any $\lambda \in (\lambda_1, \lambda_+]$, and deduce from the existence of the $\tau_{1,\tilde{c}_2^\lambda}$-subequilibrium and minimal sets a_1, $\tilde{c}_1^\lambda$ and $\tilde{c}_1^{\lambda_+}$ that any $\tilde{c}_1^\mu$ is $\tau_{1,\tilde{c}_2^\lambda}$-minimal for any $\lambda \in (\lambda_1, \lambda_+]$ and $\mu \in (0, 1]$. So that if also N exists, we are in the multiple-labeling situation, impossible. □

Let us now describe the examples of different global dynamics before mentioned, corresponding to two-dimensional autonomous ODEs $x_1' = F_1(x_1, x_2)$, $x_2' = F_2(x_1, x_2)$. The function F_1 is given by (3.2), and

$$F_2(x_1, x_2) = x_1 - \sigma_{a,b}(x_2) + c$$

for $a, b, c \in \mathbb{R}$ with $a < b$ and $c > 0$, where $\sigma_{a,b}(x_2)$ takes the values $(x_2 - a)^2$, 0 and $(x_2 - b)^2$ in the intervals $(-\infty, a)$, $[a, b]$ and (b, ∞) respectively. Note that F_2 is C^1, increasing in x_1 and concave, and that it vanishes exactly in the curve $x_1 = \sigma_{a,b}(x_2) - c$. The set $\mathcal{M}$ is given by the common zeros of F_1 and F_2. By playing with the different values of a, b and c we obtain examples on which $\mathcal{M}$ is connected (given by the whole curve $x_1 = \sigma_{a,b}(x_2) - c$) or composed by two arcs, and in this second case there are abscise points corresponding to zero, one or two equilibrium points. To construct an example in which there are infinite equilibria sharing the first coordinate, we define $F_1(x_1, x_2) = -\sigma_{a,b}(x_1)$, which satisfies the required conditions and vanishes in the strip $a \leqslant x_1 \leqslant b$, and take $a = -1$, $b = 0$ and $c = 1$ in F_1 and the above F_2: all the points $(-1, d)$ for $d \in [-1, 0]$ are equilibria.

A more sophisticated example is given by Proposition 4.10 in [27]: in that case, there exists a top minimal set, all the minimal sets are labeled on their first component, and the second one is not a multiple of the first one of the reference minimal set. A simple modification provides a sample of the analogous situation in the absence of a top minimal set.

We complete this section with three more nonautonomous examples. The omitted details can be found in Johnson [11,12], Yi [36], and Jorba et al. [15].

Example 3.13. We describe a case of dynamics (ã) for $\mathcal{M}$ for which one of the components of a semicontinuous subequilibrium a strongly below a minimal set is not continuous. Let (Ω, σ) be an almost periodic and minimal flow with unique ergodic measure m, and let $b : \Omega \to \mathbb{R}$ be a continuous function with $\|b\|_\infty = 1$ and $\int_\Omega b(\omega)\, dm = 0$, which does not admit a continuous primitive. Then there exists $(\omega_0, x_0) \in \Omega \times \mathbb{R}$ with $x_0 < 0$ such that the solution $x(t, \omega_0, x_0)$ of the scalar equation $x' = b(\omega \cdot t)x$ is non-trivial and bounded; the sections of the compact set $K^0 = \text{closure}_{\Omega \times \mathbb{R}}\{(\omega_0 \cdot t, x(t, \omega_0, x_0)) \mid t \geqslant 0\}$ reduce to $\{0\}$ for a residual subset of Ω; and the lower-semicontinuous and non continuous map $a_2 : \Omega \to (-\infty, 0]$, $\omega \mapsto \inf K_\omega$ is C^1 along the base orbits, with $a_2'(\omega) = b(\omega)a_2(\omega)$. Let us check that the map $(0, a_2(\omega))$ is a semicontinuous subequilibrium for the system $x_1' = 0$, $x_2' = F_2(\omega \cdot t, x_1, x_2) = x_1 - x_2$, which satisfies (H1) and (H2): $a_2'(\omega) = b(\omega)a_2(\omega) \leqslant -a_2(\omega) = F_2(\omega, 0, a_2(\omega))$, so that the assertion follows from Proposition 2.6(i). Finally, $\mathcal{M} = \{\{\lambda, \lambda\} \mid \lambda \in \mathbb{R}\}$.

Example 3.14. In this example, the dynamics of $\mathcal{M}$ fits case ($\tilde{\text{b}}$), and the minimal set K^0 provided by Corollary 3.11 is not a copy of the base (but an almost automorphic extension of the base), and there exists a non-continuous subequilibrium strongly below a minimal set. It is based on the well-known example of Vinograd [34] (see also Millionščikov [19,20]), whose corresponding Riccati equation we write as $x' = F(\omega \cdot t, x)$. It is known that the set of bounded solutions is a minimal set K_2^0 given by an almost automorphic extension of the base which does not reduce to a copy of the base. In addition, according to the results in Section 3 of Núñez and Obaya [24], the equation $x_2' = F(\omega \cdot t, x_2) + \lambda$ admits for $\lambda > 0$ two ordered minimal sets given by copies of the base, say $\{n_2^\lambda\} \ll \{d_2^\lambda\}$, while no minimal set exists for $\lambda < 0$. Hypothesis (H1) and (H2) are fulfilled for the two-dimensional system $x_1' = 0, x_2' = F(\omega \cdot t, x_2) + x_1$, for which the map $(0, a_2) : \Omega \to \mathbb{R}^2$ given by $a_2(\omega) = \inf(K_2^0)_\omega$ is a semicontinuous but non-continuous subequilibrium and $\mathcal{M} = \{\{0, K_2^0\}\} \cup \{\{\lambda, d_2^\lambda\}, \{\lambda, n_2^\lambda\} \mid \lambda > 0\}$.

Example 3.15. A simple modification of the previous example provides a sample of Case A1 with non-continuous initial subequilibrium: just consider the system $x_1' = x_1(1-x_1)$, $x_2' = F(\omega\cdot t, x_2) + x_1$.

4. Uncoupling in Case A2

This last section is devoted to describe a useful tool to determine the area of the phase space in which the possible minimal sets can lie: we show that in most of the dynamical situations considered in the previous section, the coefficient functions of the families of equations must satisfy some simple conditions, rather restrictive, in an area which, roughly speaking, contains most of the minimal sets, or in their whole domain in the case of analyticity. The three theorems we present can be so understood also as criteria to preclude the occurrence of a multiple-labeling situation or a single-labeling situation ($\tilde{\text{b}}$) or ($\tilde{\text{c}}$), and some cases of ($\tilde{\text{a}}$).

We establish some previous notation and properties. In the multiple-labeling situation described in Theorem 3.7, $\mathcal{F}^*$ represents the closure of the interior of $\mathcal{F}$. In the single-labeling situation described by Theorem 3.12($\tilde{\text{b}}$), we represent $\mathcal{H}_1 = \bigcup_{\omega\in\Omega}(\{\omega\}\times(\mathcal{H}_1)_\omega)$, where $(\mathcal{H}_1)_\omega$ is the order-convex hull in X of the union of the set $\bigcup_{M\in\mathcal{M}} M_\omega$ and the point $a(\omega)$; i.e., the smallest order-convex set containing the elements of all the minimal sets and the points in the graph of the initial subequilibrium.

Proposition 4.1. *Assume that the dynamics fits Theorem* 3.12($\tilde{\text{b}}$). *Then, for any $\omega\in\Omega$, the set $(\mathcal{H}_1)_\omega$ satisfies the next properties*:

(i) *If $(x_1,x_2)\in(\mathcal{H}_1)_\omega$, then there exists $\lambda\in J_*$ with $x_1 = \tilde{c}_1^\lambda(\omega)$, and if $\lambda > \inf J_*$, then either $x_2 = d_2^\lambda(\omega)$ or $x_2 \ll d_2^\lambda(\omega)$.*

(ii) *$(\tilde{c}_1^\lambda)'(\omega) \leqslant F_1(\omega, \tilde{c}_1^\lambda(\omega), x_2)$ for every $(\tilde{c}_1^\lambda(\omega), x_2)\in(\mathcal{H}_1)_\omega$.*

Proof. It follows from Theorem 3.12 that the set

$$(\mathcal{H}_1)^0_\omega = \big\{x\in X \;\big|\; x = a(\omega) \text{ or there exists } M\in\mathcal{M} \text{ with } (\omega,x)\in M\big\}$$

satisfies properties (i) and (ii). We define

$$H\big((\mathcal{H}_1)^0_\omega\big) = \big\{\alpha x + (1-\alpha)y \;\big|\; \alpha\in[0,1],\ x,y\in(\mathcal{H}_1)^0_\omega,\ x\leqslant y\big\}.$$

It is obvious that this new set contains $(\mathcal{H}_1)^0_\omega$ and satisfies the first property in (i), while the second one follows from the concavity of the map $\lambda\to d_2^\lambda(\omega)$ and Remark 3.10(2). The concavity of F_1 shows that it also satisfies (ii). We represent $(\mathcal{H}_1)^1_\omega = H((\mathcal{H}_1)^0_\omega)$. This is the starting point of a recursive procedure in which we define $(\mathcal{H}_1)^{k+1}_\omega = H((\mathcal{H}_1)^k_\omega)$. Since all these sets satisfy (i) and (ii), so does the set $\bigcup_{k\geqslant 0}(\mathcal{H}_1)^k_\omega$. And it is clear that this last set agrees with $(\mathcal{H}_1)_\omega$. □

The sets $(\mathcal{H}_2)_\omega$ and $\mathcal{H}_2$ are defined in the symmetric way if ($\tilde{\text{c}}$) holds, and they satisfy the symmetric properties.

Theorem 4.2. *Assume that the family of ODEs* (2.2) *satisfies* (H1)–(H3) *and that the corresponding dynamics above a fits Case* A2 *of Theorem* 3.8 *in* [28].

(i) *In the multiple-labeling situation described by Theorem* 3.7,

$$F_1(\omega,x_1,x_2) = h_1(\omega)\big(x_1 - a_1(\omega)\big) + a_1'(\omega) \quad \textit{and} \tag{4.1}$$

$$F_2(\omega,x_1,x_2) = h_2(\omega)\big(x_2 - a_2(\omega)\big) + a_2'(\omega) \tag{4.2}$$

in $\mathcal{L}_1^m = \{(\omega, c_1^{\alpha_1}(\omega), x_2)\in\Omega\times\mathbb{R}^2 \mid (\alpha_1,\alpha_2)\in\mathcal{F}^*,\ x_2\geqslant c_2^{\alpha_2}(\omega)\}$ and $\mathcal{L}_2^m = \{(\omega, x_1, c_2^{\alpha_2}(\omega))\in\Omega\times\mathbb{R}^2 \mid (\alpha_1,\alpha_2)\in\mathcal{F}^*,\ x_1\geqslant c_1^{\alpha_1}(\omega)\}$ respectively.

(ii) *In the single-labeling situation described by Theorem* 3.12($\tilde{\mathrm{b}}$) (*resp.* ($\tilde{\mathrm{c}}$)), (4.1) (*resp.* (4.2)) *holds in* $\mathcal{L}_1^s = \{(\omega, \tilde{c}_1^\lambda(\omega), x_2) \in \Omega \times \mathbb{R}^2 \mid x_2 \geqslant y_2$ *for an* $(\omega, \tilde{c}_1^\lambda(\omega), y_2) \in \mathcal{H}_1\}$ (*resp. in* $\mathcal{L}_2^s = \{(\omega, x_1, \tilde{c}_2^\lambda(\omega)) \in \Omega \times \mathbb{R}^2 \mid x_1 \geqslant y_1$ *for an* $(\omega, y_1, \tilde{c}_2^\lambda(\omega)) \in \mathcal{H}_2\}$).

(iii) *In the single-labeling situation described by Theorem* 3.12($\tilde{\mathrm{a}}$) *and if* $a_2 < \tilde{a}_2$ (*resp. if* $a_1 < \tilde{a}_1$), (4.1) (*resp.* (4.2)) *holds in* $\mathcal{L}_1^d = \{(\omega, c_1^\lambda(\omega), x_2) \in \Omega \times \mathbb{R}^2 \mid \lambda \in J_*,\ x_2 \geqslant \tilde{c}_2^\lambda(\omega)\}$ (*resp. in* $\mathcal{L}_2^d = \{(\omega, x_1, c_2^\lambda(\omega)) \in \Omega \times \mathbb{R}^2 \mid \lambda \in J_*,\ x_1 \geqslant \tilde{c}_1^\lambda(\omega)\}$).

If equality (4.1) (*resp.* (4.2)) *holds in the set* $\mathcal{L}_1^m$, $\mathcal{L}_1^s$ *or* $\mathcal{L}_1^d$ (*resp.* $\mathcal{L}_2^m$, $\mathcal{L}_2^s$ *or* $\mathcal{L}_2^d$), *then* $\int_\Omega h_1(\omega)\,dm = 0$ (*resp.* $\int_\Omega h_2(\omega)\,dm = 0$) *for any ergodic measure m on the base; and if, in addition,* F_1 (*resp.* F_2) *is analytical with respect to its state arguments, then* (4.1) (*resp.* (4.2)) *holds everywhere.*

Proof. (i) We begin with the multiple-labeling situation and concentrate our attention on F_1: symmetric arguments apply to F_2. Let us take $(\alpha_1, \alpha_2) \in \operatorname{Int}\mathcal{F}$, and $(\alpha_1, \beta_2) \in \mathcal{F}^*$ with $\beta_2 < \alpha_2$. Since $(c_1^{\alpha_1})'(\omega) = F_1(\omega, c_1^{\alpha_1}(\omega), x_2)$ for $x_2 = c_2^{\beta_2}(\omega)$ and $x_2 = c_2^{\alpha_2}(\omega)$, the equality holds whenever $x_2 \geqslant c_2^{\beta_2}(\omega)$. That is, $F_1(\omega, c_1^{\alpha_1}(\omega), x_2) = (c_1^{\alpha_1})'(\omega) = \alpha_1((c_1^1)'(\omega) - a_1'(\omega)) + a_1'(\omega) = h_1(\omega)(c_1^{\alpha_1}(\omega) - a_1(\omega)) + a_1'(\omega)$ for $h_1 = ((c_1^1)' - a_1')/(c_1^1 - a_1) = (\ln(c_1^1 - a_1))'$. The continuity of F_1 proves that (4.1) holds in $\mathcal{L}_1^m$, while the boundedness of $\ln(c_1^1 - a_1)$ and Birkhoff ergodic theorem prove that any ergodic average of h_1 in Ω is zero. The assertion concerning analyticity follows from the identity theorem: for each $\omega \in \Omega$ fixed, the analytic functions $F_1(\omega, x_1, x_2)$ and $h_1(\omega)(x_1 - a_1(\omega)) + a_1'(\omega)$ agree in a connected open subset of $\mathbb{R}_+^2$, and hence on the whole $\mathbb{R}^2$.

(ii) Assuming the single-labeling situation ($\tilde{\mathrm{b}}$), we will check that

$$(\tilde{c}_1^\lambda)'(\omega) = F_1\big(\omega, \tilde{c}_1^\lambda(\omega), x_2\big) \tag{4.3}$$

for every $(\omega, \tilde{c}_1^\lambda(\omega), x_2) \in \mathcal{L}_1^s$. Let us represent $\mathcal{H}_1^{\lambda,\omega} = \{y_2 \in X_2 \mid (\tilde{c}_1^\lambda(\omega), y_2) \in (\mathcal{H}_1)_\omega\}$ for $\lambda \in J_*$ and $\omega \in \Omega$. Take first $\lambda > \inf J_*$, in which case $d_2^\lambda(\omega) \in \mathcal{H}_1^{\lambda,\omega}$ and is the maximal element (for the order in X_2) in this section set. Assume also that this set contains more elements, and take $y_2 \neq d_2^\lambda(\omega)$ in it. Proposition 4.1 ensures that $y_2 \ll d_2^\lambda(\omega)$. From the increasing properties of F_1 we obtain $(\tilde{c}_1^\lambda)'(\omega) = F_1(\omega, \tilde{c}_1^\lambda(\omega), d_2^\lambda(\omega)) \geqslant F_1(\omega, \tilde{c}_1^\lambda(\omega), y_2) \geqslant (\tilde{c}_1^\lambda)'(\omega)$, and hence its concavity ensures (4.3) for any $x_2 \geqslant y_2$, which obviously includes $x_2 \geqslant d_2^\lambda(\omega)$. There are only two situations in which $\mathcal{H}_1^{\lambda,\omega}$ can reduce to $d_2^\lambda(\omega)$. The first one is the case $\lambda = \lambda_+ = \sup J_* < \infty$. If this happens, (4.3) follows from the continuity of F_1. The second one is slightly more difficult to analyze: if $\mathcal{M} = \{\{\tilde{c}_1^\lambda, c_2^1\} \mid \lambda \in J_*\}$ and $\lambda < 0$. So that let us fix such a λ_0 in this case, and take $x_2 \geqslant c_2^1(\omega)$. Since the map $l_1(\lambda) = F_1(\omega, \tilde{c}_1^\lambda(\omega), x_2) - (\tilde{c}_1^\lambda)'(\omega)$ is concave in λ and vanishes in J, we deduce that $l_1(\lambda_0) \leqslant 0$. But the increasing properties of F_1 ensure also that $l_1(\lambda_0) \geqslant F_1(\omega, \tilde{c}_1^{\lambda_0}(\omega), c_2^1(\omega)) - (\tilde{c}_1^{\lambda_0})'(\omega) = 0$. Altogether, we obtain (4.3) also in this case.

Assume finally that $\lambda_- = \inf J_* > -\infty$, in which case $\mathcal{H}_1^{\lambda_-,\omega} \supseteq (D_2^\lambda)_\omega$. This set contention is strict if another minimal set $\{\tilde{c}_1^{\lambda_-}, N_2^{\lambda_-}\}$ exists, in which case $N_2^{\lambda_-} \ll D_2^{\lambda_-} = \{d_2^{\lambda_-}\}$ and we can reason as above to obtain (4.3). On the one hand, if both sets are equal, then $\mathcal{H}_1^{\lambda_-,\omega} = \{d_2^{\lambda_-}(\omega)\}$ in a residual subset of Ω, so that (4.3) follows again from continuity at these points; and on the other hand, as seen in Corollary 3.11, any other point $z_2 \in \mathcal{H}_1^{\lambda_-,\omega}$ satisfies $z_2 \ll d_2^{\lambda_-}(\omega)$, and $(\tilde{c}_1^{\lambda_-})'(\omega) = F_1(\omega, \tilde{c}_1^{\lambda_-}(\omega), d_2^{\lambda_-}(\omega))$ since $(\tilde{c}_1^{\lambda_-}, d_2^{\lambda_-})$ is an equilibrium, so that (4.3) can be deduced as above.

Once proved equality (4.3) for every $(\omega, \tilde{c}_1^\lambda(\omega), x_2) \in \mathcal{L}_1^s$, the last lines of the multiple-labeling proof complete this one.

(iii) Let us finally consider Case ($\tilde{\mathrm{a}}$) with $a_2 < \tilde{a}_2$, which ensures $a_1 = \tilde{a}_1$. If $a_2(\omega) \ll \tilde{a}_2(\omega)$, or equivalently if $c_2^\lambda(\omega) \ll \tilde{c}_2^\lambda(\omega)$ for any $\lambda \geqslant 0$, Proposition 3.5(i) and the usual arguments of concavity and increasing character of F_1 ensure that (4.3) holds if $x_2 \geqslant c_2^\lambda(\omega)$ and $\lambda \in J$. In addition, as explained in the proof of Proposition 3.5(iii) if $a_2(\omega_0) < \tilde{a}_2(\omega_0)$ then $a_2(\omega_0 \cdot t) \ll \tilde{a}_2(\omega_0 \cdot t)$ for any $t > 1$. This fact, the minimality of Ω, and the continuity of the functions involved ensure that (4.3) holds if $x_2 \geqslant \tilde{c}_2^{\lambda}(\omega)$ and $\lambda \in J$. To extend the result to $\lambda \in J_* \cap (-\infty, 0]$ we apply the concavity and increasing character of F_1 and the fact that $\tilde{c}^{\lambda}$ is an equilibrium. The proof is completed as in the two previous cases. □

The previous arguments can be easily adapted to the delay and parabolic cases:

Theorem 4.3. *Assume that the family of delay equations* (2.3) *satisfies* (H1)–(H3) *and that the corresponding dynamics above a fits Case* A2 *of Theorem* 3.8 *in* [28].

(i) *In the multiple-labeling situation described by Theorem* 3.7,

$$\begin{aligned} F_1(\omega, x_1, x_2, x_3, x_4) &= h_1(\omega)\big(x_1 - \bar{a}_1(\omega)\big) + \bar{a}_1'(\omega) \\ &= k_1(\omega)(x_3 - \bar{a}_1\big(\omega\cdot(-1)\big) + \bar{a}_1'(\omega) \quad \textit{and} \end{aligned} \tag{4.4}$$

$$\begin{aligned} F_2(\omega, x_1, x_2, x_3, x_4) &= h_2(\omega)\big(x_2 - \bar{a}_2(\omega)\big) + \bar{a}_2'(\omega) \\ &= k_2(\omega)(x_4 - \bar{a}_2\big(\omega\cdot(-1)\big) + \bar{a}_2'(\omega) \end{aligned} \tag{4.5}$$

in $\mathcal{L}_1^m = \{(\omega, \bar{c}_1^{\alpha_1}(\omega), x_2, \bar{c}_1^{\alpha_1}(\omega\cdot(-1)), x_4) \in \Omega \times \mathbb{R}^4 \mid (\alpha_1, \alpha_2) \in \mathcal{F}^*,\ x_2 \geqslant \bar{c}_2^{\alpha_2}(\omega),\ x_4 \geqslant \bar{c}_2^{\alpha_2}(\omega\cdot(-1))\}$ *and* $\mathcal{L}_2^m = \{(\omega, x_1, \bar{c}_2^{\alpha_2}(\omega), x_3, \bar{c}_2^{\alpha_2}(\omega\cdot(-1))) \in \Omega \times \mathbb{R}^4 \mid (\alpha_1, \alpha_2) \in \mathcal{F}^*,\ x_1 \geqslant \bar{c}_1^{\alpha_1}(\omega),\ x_3 \geqslant \bar{c}_1^{\alpha_1}(\omega\cdot(-1))\}$ *respectively.*

(ii) *In the single-labeling situation described by Theorem* 3.12($\tilde{\text{b}}$) (*resp.* ($\tilde{\text{c}}$)), (4.4) (*resp.* (4.5)) *holds in* $\mathcal{L}_1^s = \{(\omega, \bar{\tilde{c}}_1^{\lambda}(\omega), x_2, \bar{\tilde{c}}_1^{\lambda}(\omega\cdot(-1)), x_4) \in \Omega \times \mathbb{R}^4 \mid x_2 \geqslant y_2(0)$ *and* $x_4 \geqslant y_2(-1)$ *for an* $(\omega, \tilde{c}_1^{\lambda}(\omega), y_2) \in \mathcal{H}_1\}$ (*resp. in* $\mathcal{L}_2^s = \{(\omega, x_1, \bar{\tilde{c}}_2^{\lambda}(\omega), x_3, \bar{\tilde{c}}_2^{\lambda}(\omega\cdot(-1))) \in \Omega \times \mathbb{R}^4 \mid x_1 \geqslant y_1(0)$ *and* $x_3 \geqslant y_1(-1)$ *for an* $(\omega, y_1, \tilde{c}_2^{\lambda}(\omega)) \in \mathcal{H}_2\}$).

(iii) *In the single-labeling situation described by Theorem* 3.12(ã) *and if* $a_2 < \tilde{a}_2$ (*resp. if* $a_1 < \tilde{a}_1$), (4.4) (*resp.* (4.5)) *holds in* $\mathcal{L}_1^d = \{(\omega, \bar{c}_1^{\lambda}(\omega), x_2, \bar{c}_1^{\lambda}(\omega\cdot(-1)), x_4) \in \Omega \times \mathbb{R}^4 \mid \lambda \in J_*,\ x_2 \geqslant \bar{\tilde{c}}_2^{\lambda}(\omega),\ x_4 \geqslant \bar{\tilde{c}}_2^{\lambda}(\omega\cdot(-1))\}$ (*resp.* $\mathcal{L}_2^m = \{(\omega, x_1, \bar{c}_2^{\lambda}(\omega), x_3, \bar{c}_2^{\lambda}(\omega\cdot(-1))) \in \Omega \times \mathbb{R}^4 \mid \lambda \in J_*,\ x_1 \geqslant \bar{\tilde{c}}_1^{\lambda}(\omega),\ x_3 \geqslant \bar{\tilde{c}}_1^{\lambda}(\omega\cdot(-1))\}$).

If equality (4.4) (*resp.* (4.5)) *holds in the set* $\mathcal{L}_1^m$, $\mathcal{L}_1^s$ *or* $\mathcal{L}_1^d$ (*resp.* $\mathcal{L}_2^m$, $\mathcal{L}_2^s$ *or* $\mathcal{L}_2^d$), *then* $\int_\Omega h_1(\omega)\,dm = 0$ (*resp.* $\int_\Omega h_2(\omega)\,dm = 0$) *for any ergodic measure m on the base*; *and if, in addition,* F_1 (*resp.* F_2) *is independent of* x_3 *or of* x_1 (*resp. of* x_4 *or of* x_2), *and it is analytical with respect to its state arguments, then one of the two equalities in* (4.4) (*resp. in* (4.5)) *holds everywhere.*

Theorem 4.4. *Assume that the family of parabolic equations* (2.4) *satisfies* (H1)–(H3) *and that the corresponding dynamics above a fits Case* A2 *of Theorem* 3.8 *in* [28].

(i) *In the multiple-labeling situation described by Theorem* 3.7,

$$F_1(\omega, v, x_1, x_2) = h_1(\omega, v)\big(x_1 - \bar{a}_1(\omega, v)\big) - d_1\Delta\bar{a}_1(\omega, v) + \bar{a}_1'(\omega, v), \tag{4.6}$$

$$F_2(\omega, v, x_1, x_2) = h_2(\omega, v)\big(x_2 - \bar{a}_2(\omega, v)\big) - d_2\Delta\bar{a}_2(\omega, v) + \bar{a}_2'(\omega, v) \tag{4.7}$$

in $\mathcal{L}_1^m = \{(\omega, v, \bar{c}_1^{\alpha_1}(\omega, v), x_2) \in \Omega \times \bar{U} \times \mathbb{R}^2 \mid (\alpha_1, \alpha_2) \in \mathcal{F}^*,\ x_2 \geqslant \bar{c}_2^{\alpha_2}(\omega, v)\}$ *and* $\mathcal{L}_2^m = \{(\omega, v, x_1, \bar{c}_2^{\alpha_2}(\omega, v)) \in \Omega \times \bar{U} \times \mathbb{R}^2 \mid (\alpha_1, \alpha_2) \in \mathcal{F}^*,\ x_1 \geqslant \bar{c}_1^{\alpha_1}(\omega, v)\}$ *respectively.*

(ii) *In the single-labeling situation described by Theorem* 3.12($\tilde{\text{b}}$) (*resp.* ($\tilde{\text{c}}$)), (4.6) (*resp.* (4.7)) *holds in* $\mathcal{L}_1^s = \{(\omega, v, \bar{\tilde{c}}_1^{\lambda}(\omega, v), x_2) \in \Omega \times \bar{U} \times \mathbb{R}^2 \mid x_2 \geqslant y_2(v)$ *for an* $(\omega, \tilde{c}_1^{\lambda}(\omega), y_2) \in \mathcal{H}_1\}$ (*resp. in* $\mathcal{L}_2^s = \{(\omega, v, x_1, \bar{\tilde{c}}_2^{\lambda}(\omega, v)) \in \Omega \times \bar{U} \times \mathbb{R}^2 \mid x_1 \geqslant y_1(v)$ *for an* $(\omega, y_1, \tilde{c}_2^{\lambda}(\omega)) \in \mathcal{H}_2\}$).

(iii) *In the single-labeling situation described by Theorem* 3.12(ã) *and if* $a_2 < \tilde{a}_2$ (*resp. if* $a_1 < \tilde{a}_1$), (4.6) (*resp.* (4.7)) *holds in* $\mathcal{L}_1^d = \{(\omega, v, \bar{c}_1^\lambda(\omega, v), x_2) \in \Omega \times \bar{U} \times \mathbb{R}^2 \mid \lambda \in J_*,\ x_2 \geqslant \bar{\bar{c}}_2^\lambda(\omega, v)\}$ (*resp.* $\mathcal{L}_2^d = \{(\omega, v, x_1, \bar{c}_2^\lambda(\omega, v)) \in \Omega \times \bar{U} \times \mathbb{R}^2 \mid \lambda \in J_*,\ x_1 \geqslant \bar{\bar{c}}_1^\lambda(\omega, v)\}$).

If equality (4.6) (*resp.* (4.7)) *holds in the set* $\mathcal{L}_1^m$, $\mathcal{L}_1^s$ *or* $\mathcal{L}_1^d$ (*resp.* $\mathcal{L}_2^m$, $\mathcal{L}_2^s$ *or* $\mathcal{L}_2^d$), *and if, in addition,* F_1 (*resp.* F_2) *is analytical with respect to its state arguments, then* (4.6) (*resp.* (4.7)) *holds everywhere.*

Remark 4.5. The previous results extend to the concave situation we are dealing with those results about uncoupling for the sublinear context described in Section 4.1 in [27]. The proofs presented here are simpler, and in fact these arguments allow to improve the result of that paper in the multiple-labeling delay case. We point out that, although in that paper the single-labeling situation is not considered, the uncoupling of one of the coefficient functions can be obtained if there exists a *single-labeling interval* (using the terms in [27]) such that the corresponding minimal sets are labeled only on one of their components. The corresponding properties are analogous to the points (ii) of the three preceding theorems, but restricted to the area of the phase space corresponding to that single-labeling interval.

5. An application to a delayed Hopfield-type neural network

As an example of the applicability of the previous results, we analyze in this section a neural network with two coupled neurons with delayed outputs and excitatory interaction among them (see Wu [35]), described by the two-dimensional system of equations

$$\begin{aligned} x_1'(t) &= -\tilde{\alpha}_1(t)x_1(t) + \tilde{w}_{12}(t) f_2\big(x_2(t-1)\big), \\ x_2'(t) &= -\tilde{\alpha}_2(t)\, x_2(t) + \tilde{w}_{21}(t) f_1\big(x_1(t-1)\big) \end{aligned} \tag{5.1}$$

for $t > 0$, with scalar almost periodic functions $\tilde{\alpha}_1 > 0$, $\tilde{\alpha}_2 > 0$, $\tilde{w}_{12} \geqslant 0$ and $\tilde{w}_{21} \geqslant 0$. We assume that the real signal or activation functions f_i belong to $C^1(\mathbb{R})$, that $f_i(0) = 0$, and that f_i are bounded, nondecreasing and concave in $[0, \infty)$. And, for every $(x_1^0, x_2^0) \in C([-1, 0], \mathbb{R}_+^2)$, $(x_1(t, x_1^0, x_2^0), x_2(t, x_1^0, x_2^0))$ will denote the solution of (5.1) with initial data $(x_1(s, x_1^0, x_2^0), x_2(s, x_1^0, x_2^0)) = (x_1^0(s), x_2^0(s))$ for every $s \in [-1, 0]$.

It is well known that this system can be included in a special type of family (2.3),

$$\begin{aligned} x_1'(t) &= -\alpha_1(\omega\cdot t)x_1(t) + w_{12}(\omega\cdot t) f_2\big(x_2(t-1)\big), \\ x_2'(t) &= -\alpha_2(\omega\cdot t)\, x_2(t) + w_{21}(\omega\cdot t) f_1\big(x_1(t-1)\big) \end{aligned} \tag{5.2}$$

for $t > 0$ and for $\omega \in \Omega$, where Ω is the common hull of the almost periodic coefficients. Now $\alpha_1, \alpha_2, w_{12}$ and w_{21} are real continuous nonnegative functions in Ω, and they give back the coefficients of (5.1) when evaluated at a particular element ω_0 of Ω (see e.g. [27]). It is immediate to check that $(0, 0)$ is a constant equilibrium, so that $\Omega \times C([-1, 0], \mathbb{R}_+^2)$ is τ-invariant. When restricted to this area, (5.2) satisfies conditions (H1) and (H2). In order to complete hypothesis (H3) we assume the existence of a strongly positive minimal set $K^1 = \{c_1^1, c_2^1\}$. This is the situation if, for instance, there exists $k > 0$ such that the constant function (k, k) is a lower solution of (5.1), which in turn occurs if $\tilde{w}_{12} > 0$, $\tilde{w}_{21} > 0$, $f_2(k)/k \geqslant \max_{t\geqslant 0}\{\tilde{\alpha}_1(t)/\tilde{w}_{12}(t)\}$ and $f_1(k)/k \geqslant \max_{t\geqslant 0}\{\tilde{\alpha}_2(t)/\tilde{w}_{21}(t)\}$: since the boundedness of f ensures that (m, m) is a strict upper solution for the same system if $m > k$ is large enough, the semiorbit of (5.2) starting at (ω_0, l, l) with $l \in [k, m]$ is bounded and strongly above $(0, 0)$.

Remark 5.1. According to the results of [21], the fact that (m, m) is a strict upper solution for (5.1) if m is large enough ensures that it defines a strong super-equilibrium for (5.2), which in turn implies that the omega-limit set of (ω_0, m, m) is a minimal set which strongly below (m, m). It is not hard to deduce the existence of a top minimal set: the omega-limit set of any point in the graph of any of these strong superequilibria. Consequently, we can assume without restriction that K^1 is the top minimal set of the semiflow.

For further purposes we point out that the existence of K^1 is not compatible with $\tilde{w}_{12} \equiv 0$, $\tilde{w}_{21} \equiv 0$, $f_1|_{[0,\infty)} \equiv 0$ or $f_2|_{[0,\infty)} \equiv 0$.

Theorem 5.2. *Under the assumed conditions, one of the following situations holds*:

(1) *There is a unique almost periodic solution* $(x_1^*(t), x_2^*(t)) \gg (0,0)$ *of* (5.1) *such that*

$$\lim_{t\to\infty} \bigl\| \bigl(x_1\bigl(t, x_1^0, x_2^0\bigr), x_2\bigl(t, x_1^0, x_2^0\bigr)\bigr) - \bigl(x_1^*(t), x_2^*(t)\bigr) \bigr\| = 0$$

whenever $(x_1^0, x_2^0) \in C([-1,0], \mathbb{R}^2)$ *and* $(x_1^0, x_2^0) \gg (0,0)$. *More precisely, given any* $\delta > 0$ *there exist* $\kappa_\delta > 1$ *and* $\rho > 0$ *such that if* $(x_1^0, x_2^0) \geqslant (\delta, \delta)$ *then, for* $t \geqslant 0$,

$$\begin{aligned} &\bigl\| \bigl(x_1\bigl(t, x_1^0, x_2^0\bigr), x_2\bigl(t, x_1^0, x_2^0\bigr)\bigr) - \bigl(x_1^*(t), x_2^*(t)\bigr) \bigr\| \\ &\quad \leqslant \kappa_\delta e^{-\rho t} \max_{s\in[-1,0]} \bigl\| \bigl(x_1^0(s), x_2^0(s)\bigr) - \bigl(x_1^*(s), x_2^*(s)\bigr) \bigr\|. \end{aligned}$$

(2) *There exists an almost periodic solution* $(x_1^*(t), x_2^*(t)) \gg (0,0)$ *of* (5.1) *such that the functions* $(\delta x_1^*(t), \delta x_2^*(t))$ *for* $\delta \in (0,1]$ *are all the strongly positive almost periodic solutions, and given any strongly positive* $(x_1^0, x_2^0) \in C([-1,0], \mathbb{R}^2)$ *there exists* $\delta \in (0,1]$ *such that*

$$\lim_{t\to\infty} \bigl\| \bigl(x_1\bigl(t, x_1^0, x_2^0\bigr), x_2\bigl(t, x_1^0, x_2^0\bigr)\bigr) - \bigl(\delta x_1^*(t), \delta x_2^*(t)\bigr) \bigr\| = 0.$$

In particular, $\delta = 1$ *if* $(x_1^0, x_2^0) \geqslant (x_1^*, x_2^*)$. *Finally, for* $i = 1$ *and* $i = 2$, *there exists a maximal* $\varepsilon_i > 0$ *such that* f_i *is a nonvanishing linear function in* $[0, \varepsilon_i]$.

Proof. Note that $(x_1^*(t), x_2^*(t)) = (c_1^1(\omega_0 \cdot t)(0), c_2^1(\omega_0 \cdot t)(0))$ is a strongly positive almost periodic solution of (5.1). The assertions in case (1) follow from [28], Theorem 3.8 (i) and (iv) in the case that the family (5.2) fits Case A1 of that result: that is, if K^1 is the unique strongly positive minimal set.

Assume that this is not the case and, by contradiction, that the family (5.2) is, with respect to K^1, in the multiple-labeling situation of Theorem 3.6 or the single-labeling situation (b) of Theorem 3.9. Then the uncoupling described by Theorem 4.3 ensures that the function $F_1(\omega, x_1, x_2, x_3, x_4) = -\alpha_1(\omega)x_1 + w_{12}(\omega) f_2(x_4)$ determining the first equation of (5.2) satisfies

$$F_1\bigl(\omega, \delta\bar{c}_1^1(\omega), x_2, \delta\bar{c}_1^1\bigl(\omega\cdot(-1)\bigr), x_4\bigr) = h_1(\omega)\delta\bar{c}_1^1(\omega)$$

if $x_2 \geqslant \delta\bar{c}_2^1(\omega)$ and $x_4 \geqslant \delta\bar{c}_2^1(\omega\cdot(-1))$ for $\delta \in (0,1]$. We fix ω with $w_{12}(\omega) \neq 0$ and deduce from the above expressions for F_1 that f_2 is constant in $[\delta c_2^1(\omega\cdot(-1)), \infty)$ for any $\delta > 0$, which by continuity implies that it is constant (and hence null) in $[0, \infty)$. As said before, this is not compatible with the assumed existence of K^1. The same argument precludes the single-labeling situation (c).

Consequently, if (1) does not hold then the dynamics fits situation (a) of Theorem 3.9 with top minimal set K^1, which means that the set of strongly positive minimal sets is composed by $\{\delta c_1^1, \delta c_2^1\}$ for $\delta \in (0,1]$. By substituting the solutions $(\delta x_1^*(t), \delta x_2^*(t))$ in (5.1) at $t = 1$ we conclude that $f_2(x) = (f_2(x_2^*(0))/x_2^*(0))x$ and $f_1(x) = (f_1(x_1^*(0))/x_1^*(0))x$ if $x > 0$ is close to 0. The boundedness of f_i ensures that this linearity just holds in a finite subinterval $[0, \varepsilon_i]$, proving the last assertion in (2). Theorem 3.8(v) in [28] and Theorem 3.1 complete the proof. □

Let us now consider the case in which one of the two neurons, say the first one, is self-connected. Now the evolution of the network is described by

$$\begin{aligned} x_1'(t) &= -\tilde{\alpha}_1(t)x_1(t) + \tilde{w}_{11}(t)f_1\big(x_1(t-1)\big) + \tilde{w}_{12}(t)f_2\big(x_2(t-1)\big), \\ x_2'(t) &= -\tilde{\alpha}_2(t)x_2(t) + \tilde{w}_{21}(t)f_1\big(x_1(t-1)\big) \end{aligned} \tag{5.3}$$

for $t > 0$, with scalar almost periodic functions $\tilde{\alpha}_1 > 0$, $\tilde{\alpha}_2 > 0$, $\tilde{w}_{11} \geqslant 0$, $\tilde{w}_{12} \geqslant 0$ and $\tilde{w}_{21} \geqslant 0$, where f_1 and f_2 are as above and $\tilde{w}_{11} \not\equiv 0$. In this case, the family over the common hull Ω of the almost periodic coefficients is given by

$$\begin{aligned} x_1'(t) &= -\alpha_1(\omega\cdot t)x_1(t) + w_{11}(\omega\cdot t)f_1\big(x_1(t-1)\big) + w_{12}(\omega\cdot t)f_2\big(x_2(t-1)\big), \\ x_2'(t) &= -\alpha_2(\omega\cdot t)x_2(t) + w_{21}(\omega\cdot t)f_1\big(x_1(t-1)\big) \end{aligned} \tag{5.4}$$

for $t > 0$ and for $\omega \in \Omega$, and system (5.3) agrees with (5.4) for a particular element $\omega_0 \in \Omega$. Again, $(0,0)$ is a constant equilibrium, and we assume the existence of a strongly positive minimal set $K^1 = \{c_1^1, c_2^1\}$. (As in the previous and next cases, it is possible to determine conditions ensuring that (k,k) is a lower solution of (5.3) for $k > 0$ small enough.) Therefore, the restriction of the semiflow defined by (5.4) to the invariant subset $\Omega \times C([-1,0],\mathbb{R}_+^2)$ satisfies hypotheses (H1), (H2) and (H3). Note that the existence of K^1 implies that $w_{21} \not\equiv 0$ and $f_1|_{[0,\infty)} \not\equiv 0$. And, as explained in Remark 5.1, we can assume without restriction that K^1 is the top minimal set.

Theorem 5.3. *Under the assumed conditions, the conclusions of Theorem* 5.2 *hold for* (5.3)*, with the only difference that f_2 can be identically zero in* $[0,\infty)$.

Proof. As in Theorem 5.2, situation (1) holds when K^1 is the unique strongly positive minimal set for the semiflow defined by (5.4), and the same argument as there shows that the multiple-labeling situation with respect to K^1 of Theorem 3.6 or the single-labeling situation (c) of Theorem 3.9 would imply $f_1|_{[0,\infty)} \equiv 0$, impossible. Thus, if (1) does not hold, the dynamics fits either (b) or (a).

Let us get a contradiction by assuming that (b) holds. The uncoupling described by Theorem 4.3 ensures that the function $F_1(\omega, x_1, x_2, x_3, x_4) = -\alpha_1(\omega)x_1 + w_{11}(\omega)f_1(x_3) + w_{12}(\omega)f_2(x_4)$ determining the first equation of (5.4) satisfies

$$F_1\big(\omega, \delta\bar{c}_1^1(\omega), x_2, \delta\bar{c}_1^1\big(\omega\cdot(-1)\big), x_4\big) = h_1(\omega)\delta\bar{c}_1^1(\omega) = k_1(\omega)\delta\bar{c}_1^1\big(\omega\cdot(-1)\big)$$

if $x_2 \geqslant \delta\bar{c}_2^1(\omega)$ and $x_4 \geqslant \delta\bar{c}_2^1(\omega\cdot(-1))$. As in the previous proof, these equalities imply that, for any $\omega \in \Omega$, $w_{12}(\omega)f_2$ is constant in $[\delta c_2^1(\omega\cdot(-1)), \infty)$ for any $\delta \in (0,1]$, and hence it vanishes in $[0,\infty)$. So that $w_{12}f_2 \equiv 0$: this term plays no role in the dynamics. In addition, since $\delta\bar{c}_1^1(\omega\cdot t)$ is a solution of the first equation of (5.4) for $\delta \in [0,1]$, it follows that $f_1(\delta\bar{c}_1^1(\omega\cdot(-1))) = \delta f_1(\bar{c}_1^1(\omega\cdot(-1)))$ and hence that $f_1(x) = (f_1(\bar{c}_1^1(\omega\cdot(-1)))/\bar{c}_1^1(\omega\cdot(-1)))x$ if $x > 0$ is close to 0. The boundedness of f_1 ensures that this linearity can only hold in a finite interval $[0,\varepsilon_1]$. In addition, since $(\bar{c}_2^1)'(\omega) = -\alpha_2(\omega)\bar{c}_2^1(\omega) + w_{21}(\omega)f_1(\bar{c}_1^1(\omega\cdot(-1)))$, then, for $\delta \in [0,1]$, $(\delta\bar{c}_2^1(\omega))' = -\alpha_2(\omega)(\delta\bar{c}_2^1(\omega)) + w_{21}(\omega)f_1(\delta\bar{c}_1^1(\omega\cdot(-1)))$. This means that all the strongly positive minimal sets are $\{\{\delta c_1^1, \delta c_2^1\} \mid \delta \in (0,1]\}$ and hence that the dynamics fits case (a), which provides the searched contradiction.

Assume finally that the dynamics fits (a). Then the same argument as in the proof of Theorem 5.2 applied to the second equation in (5.3) shows that f_1 is linear in a bounded interval $[0,\varepsilon_1]$ and hence, working now with the first equation, that this is true also for f_2. But note that in this case nothing precludes the possibility that $w_{12}f_2 \equiv 0$. The remaining assertions are proved again as in Theorem 5.2. $\square$

Note that case (2) with $w_{12} f_2 \equiv 0$ corresponds to a two-neurons network in which the second one sends no signal to the first one, which is self-connected. An easy example of this situation is given by the system $x_1'(t) = -x_1(t) + g(x_1(t-1))$, $x_2'(t) = -x_2(t) + g(x_1(t-1))$ for $t > 0$, with a concave function $g : [0, \infty) \to \mathbb{R}$ agreeing with the identity exactly in $[0, 1]$.

Let us finally consider the situation in which both neurons are self-connected:

$$\begin{aligned} x_1'(t) &= -\tilde{\alpha}_1(t)x_1(t) + \tilde{w}_{11}(t) f_1\big(x_1(t-1)\big) + \tilde{w}_{12}(t) f_2\big(x_2(t-1)\big), \\ x_2'(t) &= -\tilde{\alpha}_2(t)x_2(t) + \tilde{w}_{21}(t) f_1\big(x_1(t-1)\big) + \tilde{w}_{22}(t) f_2\big(x_2(t-1)\big) \end{aligned} \tag{5.5}$$

for $t > 0$, with scalar almost periodic functions $\tilde{\alpha}_1 > 0$, $\tilde{\alpha}_2 > 0$, $\tilde{w}_{11} \geqslant 0$, $\tilde{w}_{12} \geqslant 0$, $\tilde{w}_{21} \geqslant 0$ and $\tilde{w}_{22} \geqslant 0$. To the previous conditions on f_1 and f_2 we add their strictly positive character in $[0, \infty)$, as well as the conditions $\tilde{w}_{11} \not\equiv 0$ and $\tilde{w}_{22} \not\equiv 0$ in order to exclude the previously analyzed systems. We also assume the existence of a strongly positive minimal set K^1 for the semiflow given by the corresponding family of systems on the hull, and point out that again there is no restriction in assuming that K^1 is the top minimal set: see Remark 5.1. (This general model also describes a non self-connected four-neurons network in which the pairs (x_1, x_2) and (x_3, x_4) are synchronized.)

Theorem 5.4. *Under the assumed conditions, either there is no interaction between the two neurons, or the dynamics fits one of the four following cases*: (1) *or* (2) *of Theorem* 5.2*,*

(3) *the coefficient function $\tilde{w}_{12}$ is null, and there exists an almost periodic solution $(x_1^*(t), x_2^*(t)) \gg (0, 0)$ of* (5.5) *such that the functions $(\delta x_1^*(t), x_2^\delta(t))$ for $\delta \in (0, 1]$ are all the strongly positive almost periodic solutions, with $\delta x_2^* \ll x_2^\delta(t) \leqslant x_2^*(t)$ for $\delta \in (0, 1)$ and $x_2^1 = x_2^*$, and given any strongly positive $(x_1^0, x_2^0) \in C([-1, 0], \mathbb{R}^2)$ there exists $\delta \in (0, 1]$ such that*

$$\lim_{t\to\infty} \big\| \big(x_1\big(t, x_1^0, x_2^0\big), x_2\big(t, x_1^0, x_2^0\big)\big) - \big(\delta x_1^*(t), x_2^\delta(t)\big) \big\| = 0.$$

In particular, $\delta = 1$ if $(x_1^0, x_2^0) \geqslant (x_1^*, x_2^*)$. In addition, there exists a maximal $\varepsilon_1 > 0$ such that f_1 is a linear function in $[0, \varepsilon_1]$.

(4) *The coefficient function $\tilde{w}_{21}$ is null and the situation is symmetric to the one described in* (3)*.*

Proof. The proof uses the same arguments as the previous ones. The multiple-labeling situation of Theorem 3.7 would imply $\tilde{\omega}_{12} = \tilde{\omega}_{21} \equiv 0$; that is, the absence of interaction among neurons. If K^1 is the unique strongly positive minimal set the dynamics fits point (1) of Theorem 5.2. If the dynamics fits case (a) of Theorem 3.9, assuming that f_1 or f_2 is not linear in $[0, \varepsilon]$ for any $\varepsilon > 0$ means that it is strictly concave, and this provides a contradiction. Consequently, all the assertions in point (2) of Theorem 5.2 hold. Assume now that the dynamics fits case (b). As shown in the proof of Theorem 5.3, $\tilde{\omega}_{12} \equiv 0$ and f_1 is linear for $x \geqslant 0$ close to 0, and hence (by boundedness) exactly in an interval $[0, \varepsilon_1]$. The remaining assertions in (3) follow from Theorem 3.9 and Theorem 3.8(v) in [28]. And finally, the situation is symmetric if (4) holds. □

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